A Topological Chern-Simons Sigma Model and New Invariants of Three-Manifolds

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Abstract

We construct a topological Chern-Simons sigma model on a Riemannian three-manifold M with gauge group G whose hyperkähler target space X is equipped with a G-action. Via a perturbative computation of its partition function, we obtain *new* topological invariants of M that define *new* weight systems which are characterized by both Lie algebra structure *and* hyperkähler geometry. In canonically quantizing the sigma model, we find that the partition function on certain M can be expressed in terms of Chern-Simons knot invariants of M and the intersection number of certain G-equivariant cycles in the moduli space of G-covariant maps from M to X. We also construct supersymmetric Wilson loop operators, and via a perturbative computation of their expectation value, we obtain *new* knot invariants of M that define *new* knot weight systems which are also characterized by both Lie algebra structure *and* hyperkähler geometry.

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1. Introduction, Summary and Acknowledgements

The relevance of three-dimensional quantum field theory – in particular, topological Chern-Simons gauge theory – to the study of three-manifold invariants, was first elucidated in a seminal paper by Witten [1] in an attempt to furnish a three-dimensional interpretation of the Jones polynomial [2] of knots in three-space. Further developments [3, 4, 5, 6] along this direction culminated in the observation that certain three-manifold invariants can be expressed as weight systems whose weights depend on the Lie algebra structure which underlies the gauge group. Since these weights are naturally associated to Feynman diagrams via their relation to Chern-Simons theory, it meant that such three-manifold invariants have an alternative interpretation as Lie algebra-dependent graphical invariants. This provided a novel and interesting way to analyze them.

It was then asked if there exist other three-manifold invariants that can be expressed as weight systems whose weights depend on something else other than Lie algebra structure. This question was answered positively by Rozansky and Witten several years later in [7], where they formulated a certain three-dimensional supersymmetric topological sigma model with a hyperkähler target space – better known today as the Rozansky-Witten sigma model – and showed that one can, from its perturbative partition function, obtain such aforementioned three-manifold invariants whose weights depend not on Lie algebra structure but on hyperkähler geometry.

Naturally, one may also ask if there exist even more exotic three-manifold invariants that can be expressed as weight systems whose weights depend on both Lie algebra structure *and* hyperkähler geometry. Clearly, the quantum field theory relevant to this question ought to be a hybrid of the Chern-Simons theory and the Rozansky-Witten sigma model – a topological Chern-Simons sigma model if you will. Motivated by the formulation of such exotic three-manifold invariants among other things, the first example of a topological Chern-Simons sigma model – also known as the Chern-Simons-Rozansky-Witten (CSRW) sigma model – was constructed by Kapustin and Saulina in [8]. Shortly thereafter, a variety of other topological Chern-Simons sigma models was also constructed by Koh, Lee and Lee in [9], following which, the CSRW model was reconstructed via the AKSZ formalism by Källén, Qiu and Zabzine in [10], where a closely-related (albeit non-Chern-Simons) BF-Rozansky-Witten sigma model was also presented.

In these cited examples, the formulation and discussion of such exotic three-manifold invariants, if any at all, were rather abstract. Our main goal in this paper is to construct an appropriate Chern-Simons sigma model that would allow us to formulate and discuss, in

a concrete and down-to-earth manner accessible to most physicists, such novel and exotic three-manifold invariants, their knot generalizations, and beyond. Let us now give a brief plan and summary of the paper.

A Brief Plan and Summary of the Paper

In section 2, we construct from scratch, a topological Chern-Simons sigma model on a Riemannian three-manifold M with gauge group G whose hyperkähler target space X is equipped with a G-action, where G is a compact Lie group with Lie algebra $\mathfrak{g}$. Our model is a dynamically G-gauged version of the Rozansky-Witten sigma model, and it is closely-related to the Chern-Simons-Rozansky-Witten sigma model of Kapustin-Saulina: the Lagrangian of the models differ only by some mass terms for certain bosonic and fermionic fields. We also present a gauge-fixed version of the action, and discuss the (in)dependence of the partition function on the various coupling constants of the theory.

In section 3, we compute perturbatively the partition function of the model. This is done by first expanding the quantum fields around points of stationary phase, and then evaluating the resulting Feynman diagram expansion of the path integral without operator insertions. Apart from obtaining *new* three-manifold invariants which define *new* weight systems whose weights are characterized by both the Lie algebra structure of $\mathfrak{g}$ *and* the hyperkähler geometry of X, we also find that (i) the one-loop contribution is a topological invariant of M that ought to be related to a hybrid of the analytic Ray-Singer torsion of the flat and trivial connection on M, respectively; (ii) an "equivariant linking number" of knots in M can be defined out of the propagators of certain fermionic fields.

In section 4, we canonically quantize the time-invariant model in a neigborhood $\Sigma \times I$ of M, where Σ is an arbitrary compact Riemann surface. We find that we effectively have a two-dimensional gauged sigma model on Σ, and that the relevant Hilbert space of states would be given by the tensor product of the Hilbert space of Chern-Simons theory on M and the G-equivariant cohomology of the moduli space $\mathcal{M}^{\vartheta}$ of G-covariant maps from M to X. On three-manifolds M^U which can be obtained from M by a U-twisted surgery on $\Sigma = \mathbf{T}^2$, where U is the mapping class group of Σ, the corresponding partition function $Z_X(M^U)$ can be expressed in terms of Chern-Simons knot invariants of M and the intersection number of certain G-equivariant cycles in $\mathcal{M}^{\vartheta}$.

In section 5, we construct supersymmetric Wilson loop operators and compute perturbatively their expectation value. In doing so, we obtain *new* knot invariants of M that also define *new* knot weight systems whose weights are characterized by both the Lie algebra structure of $\mathfrak{g}$ *and* the hyperkähler geometry of X.

2. A Topological Chern-Simons Sigma Model

2.1. The Fields and the Action

We would like to construct a topological Chern-Simons (CS) sigma model that is a dynamically G-gauged version of the Rozansky-Witten (RW) sigma model on M with target space X, where M is a three-dimensional Riemannian manifold with local coordinates x^μ, $\mu = 1, 2, 3$, and X is a hyperkähler manifold of complex dimension $\dim_{\mathbb{C}} X = 2n$ which admits an action of a compact Lie group G. Let $\{V_a\}$ where $a = 1, 2, \cdots, \dim G$, be the set of Killing vector fields on X which correspond to this G-action; they can be viewed as sections of $TX \otimes \mathfrak{g}^*$, where TX is the tangent bundle of X, while $\mathfrak{g}$ is the Lie algebra of G. If we denote the local complex coordinates of X as $(\phi^I, \phi^{\bar{I}})$, where $I, \bar{I} = 1, \cdots, 2n$, one can also write these vector fields as

$$V_a = V_a^I \partial_I + V_a^{\bar{I}} \partial_{\bar{I}}.$$

Note that the V_a's satisfy the Lie algebra

$$[V_a, V_b] = f_{ab}^c V_c,$$

where the f_{ab}^c's are the structure constants of $\mathfrak{g}$. Therefore, ϕ^I and $\phi^{\bar{I}}$ must transform under the G-action as

$$\delta_\epsilon \phi^I = \epsilon^a V_a^I, \quad \delta_\epsilon \phi^{\bar{I}} = \epsilon^a V_a^{\bar{I}}.$$

In order for G to be a global symmetry of X, it is necessary and sufficient that (i) for all a, the V_a's are holomorphic or anti-holomorphic; (ii) the symplectic structure of X is preserved by the G-action associated with the V_a's. If the kähler form on X is also preserved by the G-action, locally, there would exist moment maps $\mu_+, \mu_-, \mu_3 : X \to \mathfrak{g}^*$, where

$$d\mu_{+a} = -i_{V_a}(\Omega), \quad d\mu_{-a} = -i_{V_a}(\bar{\Omega}), \quad d\mu_{3a} = -i_{V_a}(J). \qquad (2.1)$$

Here, $\Omega = \frac{1}{2}\Omega_{IJ}d\phi^I \wedge d\phi^J$ is the holomorphic symplectic form on X; $J = ig_{I\bar{K}}d\phi^I \wedge d\phi^{\bar{K}}$ is the kähler form on X; $g_{I\bar{K}}$ is the metric on X; and $i_V(\omega)$ stands for the inner product of the

vector field V with the differential form ω. The moment maps μ_+, μ_-, μ_3 are assumed to exist globally (which is automatically the case if X is simply-connected), and μ_+ is holomorphic while $\mu_- = \bar{\mu}_+$ is antiholomorphic. μ_+ also satisfies

$$\{\mu_{+a}, \mu_{+b}\} = -f_{ab}^c \mu_{+c}, \tag{2.2}$$

where the curly brackets are the Poisson brackets with respect to Ω_{IJ}. Similar formulas hold for μ_- and μ_3. We further assume that X is such that

$$\mu_+ \cdot \mu_+ = \kappa^{ab} \mu_{+a} \mu_{+b} = 0, \tag{2.3}$$

because this condition is necessary for the supersymmetry transformation defined later to be nilpotent on gauge-invariant obeservables. Note that in (2.3), κ^{ab} is the inverse of the G-invariant nondegenerate symmetric bilinear form κ_{ab} on $\mathfrak{g}$, where

$$\kappa_{ad} f_{bc}^d + \kappa_{bd} f_{ac}^d = 0. \tag{2.4}$$

Now, the fields of a G-gauged version of the RW sigma model ought to be given by

$$\text{bosonic}: \phi^I, \phi^{\bar{I}}, A_\mu^a; \quad \text{fermionic}: \eta^{\bar{I}}, \chi_\mu^I, \tag{2.5}$$

where $I, \bar{I} = 1, \cdots, 2n$; $\mu = 1, 2, 3$; and $a = 1, \cdots \dim G$. The gauge field A is a connection one-form on a principal G-bundle ε over M. With respect to an infinitesimal gauge transformation with parameter $\epsilon^a(x)$, it should transform as

$$\delta_\epsilon A^a = -(d\epsilon^a - f_{bc}^a A^b \epsilon^c) = -D\epsilon^a. \tag{2.6}$$

Since G acts on X, the bosonic fields $\phi^I, \phi^{\bar{I}}$ must be sections of a fiber bundle over M associated with ε, whose typical fiber is X. Denote this bundle as X_ε. Then, the connection A also defines a nonlinear connection on X_ε where locally, it can be thought of as a one-form on M with values in the Lie algebra of vector fields on X, i.e., $A = A^a V_a$. This means that we can write the covariant differentials of ϕ^I and $\phi^{\bar{I}}$ as

$$D\phi^I = d\phi^I + A^a V_a^I, \quad D\phi^{\bar{I}} = d\phi^{\bar{I}} + A^a V_a^{\bar{I}}.$$

As for the fermionic fields, χ_μ^I are components of a one-form χ^I on M with values in the pullback $\phi^*(T_{X_\varepsilon})$, where T_{X_ε} is the $(1,0)$ part of the fiberwise-tangent bundle of X_ε, while $\eta^{\bar{I}}$ is a zero-form on M with values in the pullback $\phi^*(\bar{T}_{X_\varepsilon})$ of the complex-conjugate bundle $\bar{T}_{X_\varepsilon}$.

From the above expressions, it is clear that the data of the Lie group G and the hyperkähler geometry of X are inextricably connected. This connection will allow us to obtain *new* three-manifold invariants which depend on both G and X, as we will show in the next section.

The Action

At any rate, let us now construct the action of the model. Let us assign to the fields ϕ, χ, η and A, the $U(1)$ R-charge $0, -1, 1$ and 0, respectively. Let us also define the following supersymmetry transformation of the fields under a scalar supercharge Q:

$$
\begin{aligned}
\delta_Q A_a &= \chi^K \partial_K \mu_{+a}, \\
\delta_Q \phi^I &= 0, \\
\delta_Q \phi^{\bar{I}} &= \eta^{\bar{I}}, \\
\delta_Q \chi^I &= D\phi^I, \\
\delta_Q \eta^{\bar{I}} &= -\bar{\xi}^{\bar{I}},
\end{aligned}
\tag{2.7}
$$

where

$$
\xi^I = V^I \cdot \mu_-, \quad \bar{\xi}^{\bar{I}} = V^{\bar{I}} \cdot \mu_+.
\tag{2.8}
$$

Here, the scalar supercharge Q is defined to have R-charge $+1$, while the moment maps $\mu_\pm$ are defined to have R-charge ± 2. Notice then that spin and R-charge are conserved in the above relations, as required.

From (2.8), we find that δ_Q^2 is a gauge transformation with parameter $\epsilon^a = -\kappa^{ab} \mu_{+b}$:

$$
\begin{aligned}
\delta_Q^2 A^a &= \kappa^{ab}(d\mu_{+b} + f_{cb}^d A^c \mu_{+d}), \\
\delta_Q^2 \phi^I &= 0, \quad \delta_Q^2 \phi^{\bar{I}} = -V^{\bar{I}} \cdot \mu_+, \\
\delta_Q^2 \chi^I &= -\chi^J \partial_J V^I \cdot \mu_+, \quad \delta_Q^2 \eta^{\bar{I}} = -\eta^{\bar{J}} \partial_{\bar{J}} V^{\bar{I}} \cdot \mu_+.
\end{aligned}
\tag{2.9}
$$

Note that to compute this, we have used $V_a^K \Omega_{KJ} V_b^J = f_{ab}^c \mu_{+c}$ and $V^I \cdot \mu_+ = 0$.

Thus, an example of a Q-invariant action S would be

$$
\begin{aligned}
S &= \int_M (L_{cs} + L_1 + L_2), \\
L_{cs} &= \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \\
L_1 &= \delta_Q(g_{I\bar{K}} \chi^I \wedge *D\phi^{\bar{K}}) \\
&= g_{I\bar{K}}(D\phi^I \wedge *D\phi^{\bar{K}} - \chi^I \wedge *D\eta^{\bar{K}}), \\
L_2 &= \frac{1}{2} \Omega_{IJ}(\chi^I \wedge D\chi^J + \frac{1}{3} R_{KL\bar{M}}^J \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}),
\end{aligned}
\tag{2.10}
$$

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where $*$ denotes the Hodge star operator on differential forms on M with respect to its Riemannian metric $h_{\mu\nu}$; 'Tr' denotes a suitably-normalized invariant quadratic form on $\mathfrak{g}$; the covariant derivatives are given by

$$D\phi^I = d\phi^I + A \cdot V^I, \quad D\chi^I = \nabla\chi^I + A \cdot \nabla_K V^I \chi^K, \quad D\eta^{\bar{I}} = \nabla\eta^{\bar{I}} + A \cdot \nabla_{\bar{K}} V^{\bar{I}} \eta^{\bar{K}};$$

∇ involves the Levi-Civita connection on X, where

$$\nabla\chi^I = d\chi^I + \Gamma^I_{JK} d\phi^J \wedge \chi^K, \quad \nabla\eta^{\bar{I}} = d\eta^{\bar{I}} + \Gamma^{\bar{I}}_{\bar{J}\bar{K}} d\phi^{\bar{J}} \wedge \eta^{\bar{K}},$$
$$\nabla_K V^I = \partial_K V^I + \Gamma^I_{KJ} V^J, \quad \nabla_{\bar{K}} V^{\bar{I}} = \partial_{\bar{K}} V^{\bar{I}} + \Gamma^{\bar{I}}_{\bar{K}\bar{J}} V^{\bar{J}};$$

and $R^J_{KL\bar{M}}$ denotes the curvature tensor of the Levi-Civita connection on X, where

$$R^J_{KL\bar{M}} = \frac{\partial \Gamma^J_{KL}}{\partial \phi^{\bar{M}}}, \quad \Gamma^I_{JK} = (\partial_J g_{K\bar{M}}) g^{I\bar{M}}.$$

2.2. Gauge-Fixing

One of our main objectives in this paper is to compute the partition function of the model. To do so, we need to gauge-fix the model. This can be done as follows.

Define the total BRST transformation

$$\delta_{\widehat{Q}} = \delta_Q + \delta_{FP},$$

where δ_{FP} is the usual Faddeev-Popov BRST operator with R-charge $+1$. The total BRST transformation $\delta_{\widehat{Q}}$ must be nilpotent, while δ_Q is nilpotent only up to a gauge transformation.

We then extend the theory by introducing fermionic Faddev-Popov ghost and anti-ghost fields c^a, $\bar{c}_a$, as well as bosonic Lagrangian multiplier fields B_a. c, $\bar{c}$, B are defined to have R-charge 1, -1 and 0, respectively. c takes values in $\mathfrak{g}$, while $\bar{c}$ and B take values in the dual Lie algebra $\mathfrak{g}^*$. By conservation of spin and R-charge, the total BRST operator $\widehat{Q}$ should

act on the fields as

$$\delta_{\widehat{Q}} A_a = dc_a - f_{abd} A^b c^d + \chi^K \partial_K \mu_{+a},$$
$$\delta_{\widehat{Q}} \phi^I = -V^I \cdot c,$$
$$\delta_{\widehat{Q}} \phi^{\bar{I}} = \eta^{\bar{I}} - V^{\bar{I}} \cdot c,$$
$$\delta_{\widehat{Q}} \chi^I = D\phi^I + (\partial_J V^{Ia}) \chi^J c_a, \qquad (2.11)$$
$$\delta_{\widehat{Q}} \eta^{\bar{I}} = \bar{\xi}^{\bar{I}} + (\partial_{\bar{J}} V^{\bar{I}a}) \eta^{\bar{J}} c_a,$$
$$\delta_{\widehat{Q}} c^a = -\kappa^{ab} \mu_{+b} + \frac{1}{2} f_{bc}^a c^b c^c,$$
$$\delta_{\widehat{Q}} \bar{c} = B,$$
$$\delta_{\widehat{Q}} B = 0.$$

It's easy to show that $\delta_{\widehat{Q}}^2 = 0$ on the fields. The $\widehat{Q}$-invariant gauge-fixed action S would then be

$$S = \int_M (L_{cs} + L_1 + L_2),$$
$$L_{cs} = \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$
$$L_1 = \delta_{\widehat{Q}} (g_{I\bar{K}} \chi^I \wedge *D\phi^{\bar{K}} + \bar{c}_a f^a) \qquad (2.12)$$
$$= g_{I\bar{K}} (D\phi^I \wedge *D\phi^{\bar{K}} - \chi^I \wedge *D\eta^{\bar{K}}) + B_a f^a - \bar{c}_a \delta_{\widehat{Q}} f^a,$$
$$L_2 = \frac{1}{2} \Omega_{IJ} (\chi^I \wedge D\chi^J + \frac{1}{3} R^J_{KL\bar{M}} \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}),$$

where f^a is some $\mathfrak{g}$-valued function; $\int_M L_{cs}$ and $\int_M L_2$ are manifestly independent of the metric of M; while $L_1 = \{\widehat{Q}, \dots\}$ is an exact form of the total BRST operator $\widehat{Q}$. Since the metric dependence of the action is of the form $\{\widehat{Q}, \dots\}$, the partition function, and also the correlation functions of $\widehat{Q}$-closed operators, are metric independent. In this sense, the theory is topologically invariant.

Notice that the transformation on the ghost field c is not standard. The standard ghost field transformation just involves the usual δ_{FP} variation, while c also gets transformed by δ_Q:

$$\delta_Q c^a = \kappa^{ab} \mu_{+b}. \qquad (2.13)$$

This fact makes the part of the action involving ghost and anti-ghost fields non-standard. For example, if we choose the Lorentz gauge $f^a = \partial^\mu A_\mu^a$, the action contains the term $\bar{c}^a \partial^\mu (\chi^K \mu_{+a})$ where the anti-ghost field $\bar{c}^a$ is coupled to the 'matter' fermion χ^K.

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2.3. About the Coupling Constants

Before we end this section, let us discuss the coupling constants of the theory as it would prove useful to do so when we carry out our computation of the partition function and beyond in the rest of the paper.

To this end, note that the partition function can be written as

$$Z = \int D\phi D\eta D\chi DADcD\bar{c}DB \, \exp\left(-\int_M (k_{cs}L_{cs} + k_1L_1 + k_2L_2)\sqrt{h}d^3x\right), \qquad (2.14)$$

where k_1, k_2 and k_{cs} are the possible coupling constants of the theory. As

$$\frac{\delta Z}{\delta k_1} = \langle \delta_{\hat{Q}}\mathcal{O} \rangle = 0, \qquad (2.15)$$

the partition function should not depend on k_1.

Let us now rescale the fields as follows:

$$\eta \to \lambda\eta, \quad \chi \to \lambda^{-1}\chi, \quad \bar{c} \to \lambda\bar{c}, \quad c \to \lambda^{-1}c, \qquad (2.16)$$

whence

$$k_1L_1 \to k_1L_1, \quad k_2L_2 \to \lambda^{-2}k_2L_2. \qquad (2.17)$$

As the field rescaling should not change the theory, the partition function should not depend on k_2 either. Thus, let us just write

$$k_1 = k_2 = k. \qquad (2.18)$$

That being said, our partition function *does* depend on the coupling constant k_{cs}. Moreover, because of the requirement of gauge invariance [1], k_{cs} ought to be quantized as

$$k_{cs} = \frac{m}{2\pi}; \quad m = 1, 2, 3 \ldots \qquad (2.19)$$

Hence, we have *two* physically distinct coupling constants in our theory. This should come as no surprise since our theory is actually a combination of a Schwarz- and Witten-type topological field theory.

3. The Perturbative Partition Function and New Three-Manifold Invariants

3.1. The Perturbative Partition Function

Let us now proceed to discuss the partition function of the gauged sigma model in the perturbative limit. To this end, recall from the last section that the partition function depends on the coupling k_{cs}. Hence, the perturbative limit of the (CS part of the) model is the same as its large k_{cs} limit. Moreover, because the partition function is independent of k, we can choose $k_1 = k_2 = k$ as large as we want. Altogether, this means that the perturbative partition function would be given by a sum of contributions centered around the points of stationary phase characterized by

$$\frac{\delta L_{cs}}{\delta A} = dA + [A, A] = 0, \tag{3.1}$$

which are the *flat connections*, and

$$\frac{\delta S}{\delta \phi} = 0 \rightarrow D^\mu \phi = 0, \tag{3.2}$$

which are the *covariantly constant* maps from M to X.

Thus, where the perturbative partition function is concerned, we can expand the gauge field A around the flat connection A_0^ϑ as

$$A_\mu^a(x) = A_{0\mu}^{\vartheta a}(x) + \tilde{A}_\mu^a(x), \tag{3.3}$$

and the bosonic scalar fields ϕ around the covariantly constant map ϕ_0 as

$$\phi^I(x) = \phi_0^I(x) + \varphi^I(x), \quad \phi^{\bar{I}}(x) = \phi_0^{\bar{I}}(x) + \varphi^{\bar{I}}(x), \tag{3.4}$$

where

$$D^\mu \phi_0^I = \partial^\mu \phi_0^I + A_0^{\vartheta a \mu} V_a^I(\phi_0) = 0, \qquad D^\mu \phi_0^{\bar{I}} = \partial^\mu \phi_0^{\bar{I}} + A_0^{\vartheta a \mu} V_a^{\bar{I}}(\phi_0) = 0. \tag{3.5}$$

Note that (3.3) means that we can write

$$L_{cs} = L_{cs}(A_0^\vartheta) + \tilde{A} \wedge d\tilde{A} + \tilde{A} \wedge [A_0^\vartheta, \tilde{A}] + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A}. \tag{3.6}$$

Let $\mathcal{M}^\vartheta$ be the space of physically distinct ϕ_0's which satisfy (3.5) for some flat connection A_0^ϑ. Assuming that the flat connection A_0^ϑ is isolated,[1] we can then write our perturbative partition function as

$$Z = k^{2n} \sum_{A_0^\vartheta} \left(e^{-\int_M L_{cs}(A_0^\vartheta)} \int_{\mathcal{M}^\vartheta} \prod_{I=1}^{2n} d\phi_0^I \prod_{\bar{I}=1}^{2n} d\phi_0^{\bar{I}} \int D\varphi D\chi D\eta D\tilde{A} Dc D\bar{c} DB \, e^{-S_{A_0^\vartheta, \phi_0}} \right). \tag{3.7}$$

---

[1] This would indeed be the case if $H^1(M, E) = 0$, where E is a flat bundle determined by A_0.

Here, k^{2n} is the normalization factor carried by the $2n$ bosonic zero modes ϕ_0, and $\int_M L_{cs}(A_0^\vartheta)+$ $S_{A_0^\vartheta,\phi_0}$ is the total action expanded around A_0^ϑ and ϕ_0.

In the total action expanded around the flat gauge field A_0^ϑ and the covariantly constant bosonic scalar fields $\phi_0^{I,\bar{I}}$, we have

$$
\begin{aligned}
D_\mu \phi^I &= \partial_\mu \phi_0^I + \partial_\mu \varphi^I + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)\{V_a^I(\phi_0) + \varphi^J \partial_J V_a^I(\phi_0) + \frac{\varphi^J \varphi^K}{2}\partial_J \partial_K V_a^I(\phi_0) + \cdots\} \\
&= \partial_\mu \varphi^I + A_{0\mu}^{\vartheta a}\varphi^J \partial_J V_a^I + \tilde{A}_\mu^a V_a^I + \tilde{A}_\mu^a \varphi^J \partial_J V_a^I + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)(\frac{\varphi^J \varphi^K}{2}\partial_J \partial_K V_a^I) + \cdots ,
\end{aligned}
\tag{3.8}
$$

since $D_\mu \phi_0^I = \partial_\mu \phi_0^I + A_{0\mu}^{\vartheta a} V_a^I(\phi_0) = 0$. We also have

$$
\begin{aligned}
D_\mu \chi_\nu^I &= \partial_\mu \chi_\nu^I + \partial_\mu \phi^J \Gamma_{JK}^I \chi_\nu^K + A_\mu^a \nabla_J V_a^I \chi_\nu^J \\
&= \partial_\mu \chi_\nu^I + \Gamma_{JK}^I D_\mu \phi^J \chi_\nu^K + A_\mu^a \partial_J V_a^I \chi_\nu^J \\
&= \partial_\mu \chi_\nu^I + \Gamma_{JK}^I D_\mu \phi^J \chi_\nu^K + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)(\partial_J V_a^I + \partial_K \partial_J V_a^I \varphi^K + \cdots)\chi_\nu^J,
\end{aligned}
\tag{3.9}
$$

where $D_\mu \phi^J$ is as given in (3.8). Similarly, one can compute the expansion of $D_\mu \eta^{\bar{I}}$.

Because of (3.8) and (3.9), we can rewrite our Lagrangian as

$$
L = \text{quadratic part} + \text{vertices}, \tag{3.10}
$$

where

$$
\text{quadratic part} = k \left(\varphi^I \ \varphi^{\bar{I}} \ \tilde{A}_\mu^a \right) \cdot L_{\text{boson}} \cdot \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}_\rho^b \end{pmatrix} + k(\eta^{\bar{I}} \ \chi_\mu^I \ \bar{c}^a \ c^a) \cdot L_{\text{fermion}} \cdot \begin{pmatrix} \eta^{\bar{J}} \\ \chi_\varrho^J \\ \bar{c}^b \\ c^b \end{pmatrix},
\tag{3.11}
$$

$$
\begin{aligned}
\text{vertices} =& \frac{k_{cs}}{3}\epsilon^{\mu\nu\rho}f_{abc}\tilde{A}_\mu^a \tilde{A}_\nu^b \tilde{A}_\rho^c \\
&-\frac{k}{2}g_{I\bar{K}}\left(\Gamma_{\bar{M}\bar{N}}^{\bar{K}}\partial_\mu \varphi^{\bar{M}}\chi^{I\mu}\eta^{\bar{N}} + \nabla_{\bar{P}}V^{\bar{K}a}\eta^{\bar{P}}\chi_\mu^I \tilde{A}_a^\mu \right) \\
&+\frac{k}{2\sqrt{h}}\Omega_{IJ}\epsilon^{\mu\nu\rho}\left(\Gamma_{MN}^J \partial_\nu \varphi^M \chi_\mu^I \chi_\rho^N + \nabla_P V^{Ja}\chi_\mu^I \chi_\rho^P \tilde{A}_{\nu a} \right) \\
&+\frac{k}{6\sqrt{h}}\Omega_{IJ}\epsilon^{\mu\nu\rho}R_{KL\bar{M}}^J \chi_\mu^I \chi_\nu^K \chi_\rho^L \eta^{\bar{M}} + f_{abd}\left(\bar{c}^a \partial^\mu \tilde{A}_\mu^b c^d + \bar{c}^a \tilde{A}_\mu^b \partial^\mu c^d \right) \\
&-\frac{k}{2}\partial_K \Gamma_{I\bar{M}\bar{N}}\partial_\mu \varphi^{\bar{M}}\varphi^K \chi^{I,\mu}\eta^{\bar{N}} + \cdots .
\end{aligned}
\tag{3.12}
$$

Here,

$$
L_{\text{boson}} = \begin{pmatrix} 0 & L_{I\bar{J}}^{\varphi\varphi} & -\frac{1}{2}V_{Ib}\partial^\rho + A_{0b}^{\vartheta\rho}V_{Ka}\partial_I V^{Kb} \\ L_{\bar{I}J}^{\varphi\varphi} & 0 & -\frac{1}{2}V_{\bar{I}b}\partial^\rho + A_{0b}^{\vartheta\rho}V_{\bar{K}a}\partial_{\bar{I}}V^{\bar{K}b} \\ \frac{1}{2}V_{Ja}\partial^\mu + A_{0b}^{\vartheta\mu}V_{Ka}\partial_J V^{Kb} & \frac{1}{2}V_{\bar{J}a}\partial^\mu + A_{0b}^{\vartheta\mu}V_{\bar{K}a}\partial_{\bar{J}}V^{\bar{K}b} & V_{Ka}V_b^K \delta^{\mu\rho} + \frac{k_{cs}}{k}\epsilon^{\mu\nu\rho}(\kappa_{ab}\partial_\nu + \frac{1}{3}f_{adb}A_{0\nu}^{\vartheta d}) \end{pmatrix},
$$

where

$$L_{I\bar{J}}^{\varphi\varphi} := \frac{1}{2}(g_{I\bar{J}}\partial^2 + g_{K\bar{L}}A_{0\mu}^{\vartheta a}A_0^{\vartheta\mu b}\partial_I V_a^K \partial_{\bar{J}} V_b^{\bar{L}}) + g_{P\bar{J}}\partial_I V_a^P A_{0\mu}^{\vartheta a}\partial^\mu,$$

$$L_{\bar{I}J}^{\varphi\varphi} := \frac{1}{2}(g_{\bar{I}J}\partial^2 + g_{K\bar{L}}A_{0\mu}^{\vartheta a}A_0^{\vartheta\mu b}\partial_J V_a^K \partial_{\bar{I}} V_b^{\bar{L}}) + g_{J\bar{P}}\partial_{\bar{I}} V_a^{\bar{P}} A_{0\mu}^{\vartheta a}\partial^\mu; \qquad (3.14)$$

$$L_{\text{fermion}} = \begin{pmatrix} 0 & -\frac{1}{2}(g_{\bar{I}J}\partial^\rho + g_{J\bar{K}}\partial_{\bar{I}}V_a^{\bar{K}}A_0^{\vartheta a\rho}) & 0 & 0 \\ \frac{1}{2}(g_{I\bar{J}}\partial^\mu + g_{I\bar{K}}\partial_{\bar{J}}V^{\bar{K}a}A_{0a}^{\vartheta\mu}) & \frac{\Omega_{IJ}}{2\sqrt{h}}\epsilon^{\mu\nu\rho}\partial_\nu + \frac{\Omega_{IK}}{2\sqrt{h}}\epsilon^{\mu\nu\rho}\partial_J V_a^K A_{0\nu}^{\vartheta a} & -\frac{1}{2}\partial_I\mu_{+b}\partial^\mu & 0 \\ 0 & \frac{1}{2}\partial_J\mu_{+a}\partial^\rho & 0 & -\frac{1}{2}\delta_b^a\partial^2 \\ 0 & 0 & \frac{1}{2}\delta_b^a\partial^2 & 0 \end{pmatrix};$$

$$(3.15)$$

the terms g_{IK} and $\Gamma_{\bar{M}\bar{N}}^{\bar{K}}$ are evaluated at ϕ_0; and $\cdots$ are other terms expanded around ϕ_0. Also notice that we ignore the multiplier field B in L_{boson}, for it would just be integrated out to give the gauge-fixing condition $f^a = 0$, where we have chosen the gauge

$$f^a = \partial^\mu A_\mu^a = 0. \qquad (3.16)$$

We can further separate the integration over the fermion zero modes η_0 and χ_0 in the path-integral and write[2]

$$Z = k^{2n} \sum_{A_0^\vartheta} \left(e^{-\int_M L_{cs}(A_0^\vartheta)} \int_{\mathcal{M}^\vartheta} \prod_{I=1}^{2n} d\phi_0^I \prod_{\bar{I}=1}^{2n} d\phi_0^{\bar{I}} \prod_{\bar{I}_i=1}^{2nb_0'} \prod_{I_j=1}^{2nb_1'} \int d\eta_0^{\bar{I}_i} \int d\chi_{0\mu}^{I_j} \int D\varphi D\tilde{\chi}D\tilde{\eta}D\tilde{A}DcD\bar{c}\, e^{-S_{A_0^\vartheta,\phi_0}} \right),$$

$$(3.17)$$

where b_0' and b_1' denote the number of fermionic zero modes $\eta_0^{\bar{I}}$ and χ_0^I, respectively; $\tilde{\eta}$ and $\tilde{\chi}$ are the corresponding *nonzero* modes; and k^{2n} is the normalization factor carried by the bosonic zero modes. One should note that the fermionic zero modes $\eta_0^{\bar{I}}$ and χ_0^I are no longer harmonic forms on M like in RW theory; this is because in our case, the kinetic operator of the fermionic fields L_{fermion} in (3.15) is no longer the Laplacian operator but a covariant version thereof. In the limit $A \to 0$, b_0', b_1' become the respective Betti numbers of M, while (3.17) becomes the partition function of the RW theory.

3.2. One-Loop Contribution

As usual, the one-loop contribution to the perturbative partition function is given by

$$Z_0 = \int D\varphi D\tilde{\chi}D\tilde{\eta}DcD\bar{c}\, e^{-S_0}, \qquad (3.18)$$

---

[2]Here, in addition to footnote 1, we assume that $H^0(M, E) = 0$ so that there are no zero modes for the ghost fields c, $\bar{c}$ and B.

where S_0 is quadratic in the fluctuating bosonic fields $\{\tilde{A}_a^\mu, \varphi^i(x)\}$ and the fermionic *nonzero* modes $\{\tilde{\eta}^I, \tilde{\chi}_\mu^I\}$:

$$S_0 = \int_M k(\,\varphi^I\ \varphi^{\bar{I}}\ \tilde{A}_\mu^a\,) \cdot L_{\text{boson}} \cdot \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}_\rho^b \end{pmatrix} + k(\,\tilde{\eta}^{\bar{I}}\ \tilde{\chi}_\mu^I\ \bar{c}^a\ c^a\,) \cdot L_{\text{fermion}} \cdot \begin{pmatrix} \tilde{\eta}^{\bar{J}} \\ \tilde{\chi}_\rho^J \\ \bar{c}^b \\ c^b \end{pmatrix}. \tag{3.19}$$

Here, the tensors $g_{I\bar{J}}, \Omega_{IJ}$ and Γ_{JK}^I which appear in L_{boson} and L_{fermion} are evaluated at some ϕ_0 in $\mathcal{M}^\vartheta$.

To compute (3.18), we first diagonalize L_{boson} and L_{fermion}:

$$P_B^T \cdot L_{\text{boson}} \cdot P_B = L'_{\text{boson}}, \tag{3.20}$$

$$P_F^T \cdot L_{\text{fermion}} \cdot P_F = L'_{\text{fermion}},$$

where L'_{boson} and L'_{fermion} are diagonal matrices, and P_B and P_F are orthonormal matrices $(P^T = P^{-1})$ constructed from the eigenvectors of L_{boson} and L_{fermion}. Because $PP^T = 1$, we can rewrite S_0 as

$$S_0' = \int_M k(\,\varphi'^I\ \varphi'^{\bar{I}}\ \tilde{A}_\mu'^a\,) \cdot L'_{\text{boson}} \cdot \begin{pmatrix} \varphi'^J \\ \varphi'^{\bar{J}} \\ \tilde{A}_\rho'^b \end{pmatrix} + k(\,\tilde{\eta}'^{\bar{I}}\ \tilde{\chi}_\mu'^I\ \bar{c}'^a\ c'^a\,) \cdot L'_{\text{fermion}} \cdot \begin{pmatrix} \tilde{\eta}'^{\bar{J}} \\ \tilde{\chi}_\rho'^J \\ \bar{c}'^b \\ c'^b \end{pmatrix}, \tag{3.21}$$

where

$$\begin{pmatrix} \varphi'^J \\ \varphi'^{\bar{J}} \\ \tilde{A}_\rho'^b \end{pmatrix} := P_B^T \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}_\rho^b \end{pmatrix}, \tag{3.22}$$

and

$$\begin{pmatrix} \tilde{\eta}'^{\bar{J}} \\ \tilde{\chi}_\rho'^J \\ \bar{c}'^b \\ c'^b \end{pmatrix} := P_F^T \begin{pmatrix} \tilde{\eta}^{\bar{J}} \\ \tilde{\chi}_\rho^J \\ \bar{c}^b \\ c^b \end{pmatrix}. \tag{3.23}$$

Moreover, the Jocabian determinants

$$\det(P) = \det(P^T) = 1 \tag{3.24}$$

for both the bosonic and fermionic fields. Therefore, the measure of the path integral is such that

$$D\varphi D\tilde{\chi} D\tilde{\eta} Dc D\bar{c} = D\varphi' D\tilde{\chi}' D\tilde{\eta}' Dc' D\bar{c}'. \tag{3.25}$$

In all, this means that the one-loop partition function can be rewritten as

$$Z_0 = \int D\varphi' D\tilde{\chi}' D\tilde{\eta}' Dc' D\bar{c}' e^{S_0'}. \tag{3.26}$$

Now because L'_{boson} and L'_{fermion} are diagonal matrices, the path integral becomes a Gaussian integral which can be directly computed as

$$Z_0 = \left(\frac{\det'' L'_{\text{fermion}}}{\det'' L'_{\text{boson}}} \right)^{\frac{1}{2}}, \tag{3.27}$$

where by using (3.20) and $\det(P^T P) = 1$, we finally get

$$Z_0 = \left(\frac{\det'' L_{\text{fermion}}}{\det'' L_{\text{boson}}} \right)^{\frac{1}{2}}. \tag{3.28}$$

Here, the superscript $''$ indicates that only nonzero modes are considered, and L_{fermion} and L_{boson} are explicitly given by (3.15) and (3.13), respectively.

As discussed in [4], the (magnitude of the) one-loop contribution to the perturbative partition function of CS theory on M corresponds to the analytic Ray-Singer torsion of the flat connection on M, while the (magnitude of the) one-loop contribution to the perturbative partition function of RW theory on M corresponds to the analytic Ray-Singer torsion of the trivial connection on M. Since our theory is a combination of both these theories, (the magnitude of) Z_0 ought to be related to a hybrid of these aforementioned topological invariants of M.

3.3. The Vacuum Expectation Value of Fermionic Zero Modes

Notice that we may call the zero modes $\chi^I_{0\mu}$ and $\eta^{\bar{I}}_0$ of the covariant Laplacian operator L_{fermion}, covariant harmonic one- and zero-forms on M with values in the tangent and complex-conjugate tangent fibers $V_{\phi_0(x)}$ and $\bar{V}_{\phi_0(x)}$ over $\mathcal{M}^\vartheta$ evaluated at the covariantly constant map $\phi_0(x)$. Because

$$\#(\text{zero modes of } \eta^{\bar{I}}) = 2n \times b'_0, \tag{3.29}$$
$$\#(\text{zero modes of } \chi^I_\mu) = 2n \times b'_1,$$

only a product of $2nb'_0$ fields $\eta^{\bar{I}}_0$ with $2nb'_1$ fields $\chi^I_{0\mu}$ has a nonzero vacuum expectation value.

Notice also that the self-products of $\eta^{\bar{I}}_0$ and $\chi^I_{0\mu}$ are elements of the space

$$H_\eta = \wedge^{\max}(\Omega'^0(M) \otimes \bar{V}_{\phi_0(x)}), \tag{3.30}$$
$$H_\chi = \wedge^{\max}(\Omega'^1(M) \otimes V_{\phi_0(x)}),$$

where $\Omega'^i(M)$ is the space of covariant harmonic i-forms on M. There is a lattice inside $\Omega'^1(M)$ which is formed by covariant harmonic one-forms with integer-valued integrals over dual one-cycles in M; let $\omega_\mu^{(\alpha)}$, where $1 \leq \alpha \leq b_1'$, be a basis of this lattice. Then, a natural measure for the fermion zero modes can be defined by normalizing the fermionic vacuum expectation values as

$$\langle \eta_0^{\bar{I}_1}(x_1) \cdots \eta_0^{\bar{I}_{2nb_0'}}(x_{2nb_0'}) \rangle = k^{-nb_0'} \epsilon^{\bar{I}_1 \cdots \bar{I}_{2nb_0'}} := \frac{k^{-nb_0'}}{2n} \sum_{s \in S_{2nb_0'}} (-1)^{|s|} \epsilon^{\bar{I}_{s(1)} \bar{I}_{s(2)}} \cdots \epsilon^{\bar{I}_{s(2nb_0'-1)} \bar{I}_{s(2nb_0')}},$$

(3.31)

and

$$\langle \chi_{0\mu_1}^{I_1}(x_1) \cdots \chi_{0\mu_{2nb_1'}}^{I_{2nb_1'}}(x_{2nb_1'}) \rangle = \frac{k^{-nb_1'}}{((2n)!)^{b_1'}} \sum_{s \in S_{2nb_1'}} (-1)^{|s|} \times$$

$$\prod_{\alpha=0}^{b_1'-1} \left(\epsilon^{I_{s(2\alpha n+1)} \cdots I_{s(2\alpha n+2n)}} \omega_{\mu_{s(2\alpha n+1)}}^{(\alpha)}(x_{s(2\alpha n+1)}) \cdots \omega_{\mu_{s(2\alpha n+2n)}}^{(\alpha)}(x_{s(2\alpha n+2n)}) \right),$$

(3.32)

say in the Feynman diagrams associated with the computation of the perturbative partition function, where S_m is the symmetric group of m elements, and $|s|$ is the parity of a permutation s.

3.4. Feynman Diagrams

Let us now analyze the Feynman diagrams associated with the computation of the perturbative partition function. Note that all diagrams which contribute to the partition function should have (i) the right number of fermionic zero modes in the corresponding vertices to absorb those that appear in the path integral measure; (ii) a k^{-2n} factor for canceling the normalization factor k^{2n} that accompanies the partition function in (3.17), because the partition function should be independent of the coupling constant k.

In RW theory [7], only a finite number of diagrams contribute to the partition function after (i) and (ii) are satisfied. In our case however, because we have, in our action, a Chern-Simons part with coupling constant $k_{cs} \neq k$, there would be an infinite number of diagrams contributing to our partition function. Fortunately though, the analysis is still tractable whence we would be able to derive some very insightful and concrete formulas in the end, as we shall see.

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Canceling the Normalization Factor of k^{2n}

At any rate, before we proceed to say more about the Feynman diagrams, let us discuss how one can cancel the aforementioned normalization factor of k^{2n}. To this end, first note that in the CS part of the action, the gauge field has quadratic term

$$k_{cs}AD^0A = k_{cs}\epsilon^{\mu\nu\rho}A_\mu^a(\kappa_{ab}\partial_\rho + \frac{1}{3}f_{adb}A_{0\rho}^{\vartheta d})A_\nu^b. \tag{3.33}$$

Therefore, the propagator of the gauge field is *a priori*

$$\triangle^{A_\mu A_\nu} \sim \frac{1}{k_{cs}}. \tag{3.34}$$

However, upon expanding the Lagrangian around A_0 and ϕ_0, the gauge field will acquire a mass term

$$kV_{Ka}V_b^K A^{\nu a}A_\nu^b. \tag{3.35}$$

As such, the propagator would become

$$\triangle^{A_\mu A_\nu} = (k_{cs}D^0 + kV_K \cdot V^K)^{-1}. \tag{3.36}$$

That being said, because the partition function does not depend on k, we can choose

$$\frac{k_{cs}}{k} \gg 1. \tag{3.37}$$

In turn, this means from (3.36) that

$$\triangle^{A_\mu A_\nu} \sim \frac{1}{k_{cs}}. \tag{3.38}$$

Hence, in what follows, we will note that $\triangle^{A_\mu A_\nu} \sim k_{cs}^{-1}$, while the other propagators are $\sim k^{-1}$.

Now, let us consider a diagram with V vertices, emanating L legs. Assume that this diagram contains V_{cs} vertices $\frac{k_{cs}}{3}A \wedge A \wedge A$ which therefore contribute a factor of $k_{cs}^{V_{cs}}$; all the other $V - V_{cs}$ vertices therefore contribute a factor of $k^{V-V_{cs}}$. Let L_{cs} be the total number of legs which are joined together by the propagator $\triangle^{A_\mu A_\nu}$, where $\mu \neq \nu$; they contribute a factor of $k_{cs}^{-\frac{L_{cs}}{2}}$. As the other propagators carry a factor of k^{-1}, while each fermionic zero mode carries a normalization factor of $k^{-\frac{1}{2}}$, the remaining $L - L_{cs}$ legs contribute a factor of $k^{-\frac{L-L_{cs}}{2}}$. Thus, this diagram contains a factor of

$$k^{-\left(\frac{L-L_{cs}}{2} - (V - V_{cs})\right)}, \tag{3.39}$$

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but because the partition function is independent of k, it must be that

$$\frac{L - L_{cs}}{2} - (V - V_{cs}) = 2n. \tag{3.40}$$

In other words, our diagrams must obey (3.40) so that the normalization factor of k^{2n} can be cancelled out.

Notice that in the case where $A \to 0$ whence $L_{cs} = V_{cs} = 0$ and our model reduces to the RW model, (3.40) would coincide with [7, eqn. (3.25)], as expected.

The Structure of the Feynman Diagrams

Note that although the computation of the partition function involves summing an infinite number of Feynman diagrams because there is no constraint on k_{cs}, one can actually classify the vertices they involve into three types.

(1) The pure gauge field vertex coming from the CS interaction

$$k_{cs} f_{abc} \epsilon^{\mu\nu\rho} A^a_\mu A^b_\nu A^c_\rho. \tag{3.41}$$

(2) The vertices free of gauge fields, such as

$$k\Omega_{IJ} \epsilon^{\mu\nu\rho} \Gamma^J_{MN} \partial_\nu \varphi^M \chi^I_\mu \chi^N_\rho \qquad \text{and} \qquad k\Omega_{IJ} \epsilon^{\mu\nu\rho} R^J_{KL\bar{M}} \chi^I_\mu \chi^K_\nu \chi^L_\rho \eta^{\bar{M}}. \tag{3.42}$$

(3) The vertices that mix matter fields[3] with gauge fields, such as

$$k\Omega_{IJ} \epsilon^{\mu\nu\rho} \nabla_P V^{Ja} \chi^I_\mu \chi^P_\rho A_{\nu a} \qquad \text{and} \qquad k g_{I\bar{K}} \nabla_{\bar{P}} V^{\bar{K}a} \eta^{\bar{P}} \chi^I_\mu A^\mu_a. \tag{3.43}$$

These three types of vertices are illustrated in Fig. 1.

3.5. The Propagator Matrices and an Equivariant Linking Number of Knots

In order to compute the Feynman diagrams, one would also need to have a knowledge of the propagators of the bosonic and fermionic fields associated with the kinetic operators L_{boson} and L_{fermion}.

The propagator of the bosonic fields $\triangle_{\text{boson}}$ can be obtained by solving the equation

$$k L_{\text{boson}(IK,\bar{I}\bar{K},ad,\mu\rho)}(x) \times \triangle^{(KJ,\bar{K}\bar{J},db,\rho\nu)}_{\text{boson}}(x-y) = \begin{pmatrix} \delta^I_J & 0 & 0 \\ 0 & \delta^{\bar{I}}_{\bar{J}} & 0 \\ 0 & 0 & \delta^\mu_\nu \delta^a_b \end{pmatrix} \cdot \delta(x-y), \tag{3.44}$$

---

[3]Here, for convenience, we use "matter fields" to mean ϕ, χ and η.

$type(1)$

μ, a

ρ, c

$k_{cs} f_{abc} \epsilon^{\mu\nu\rho}$

ν, b

A_a^μ

φ^I

$\varphi^{\bar{I}}$

χ_μ^I

$\eta^{\bar{I}}$

$type(2)$

I, μ

p^ν

M

$k\Omega_{IJ} \Gamma^J_{MN} \epsilon^{\mu\nu\rho}$

N, ρ

I, μ

L, ρ

$k\Omega_{IJ} R^J_{KL\bar{M}} \epsilon^{\mu\nu\rho}$

$\bar{M}$

K, ν

$type(3)$

I, μ

$k\Omega_{IJ} \nabla_P V^{Ja} \epsilon^{\mu\nu\rho}$

ν, a

P, ρ

I, μ

$kg_{I\bar{K}} \nabla_{\bar{P}} V^{\bar{K}a}$

μ, a

$\bar{P}$
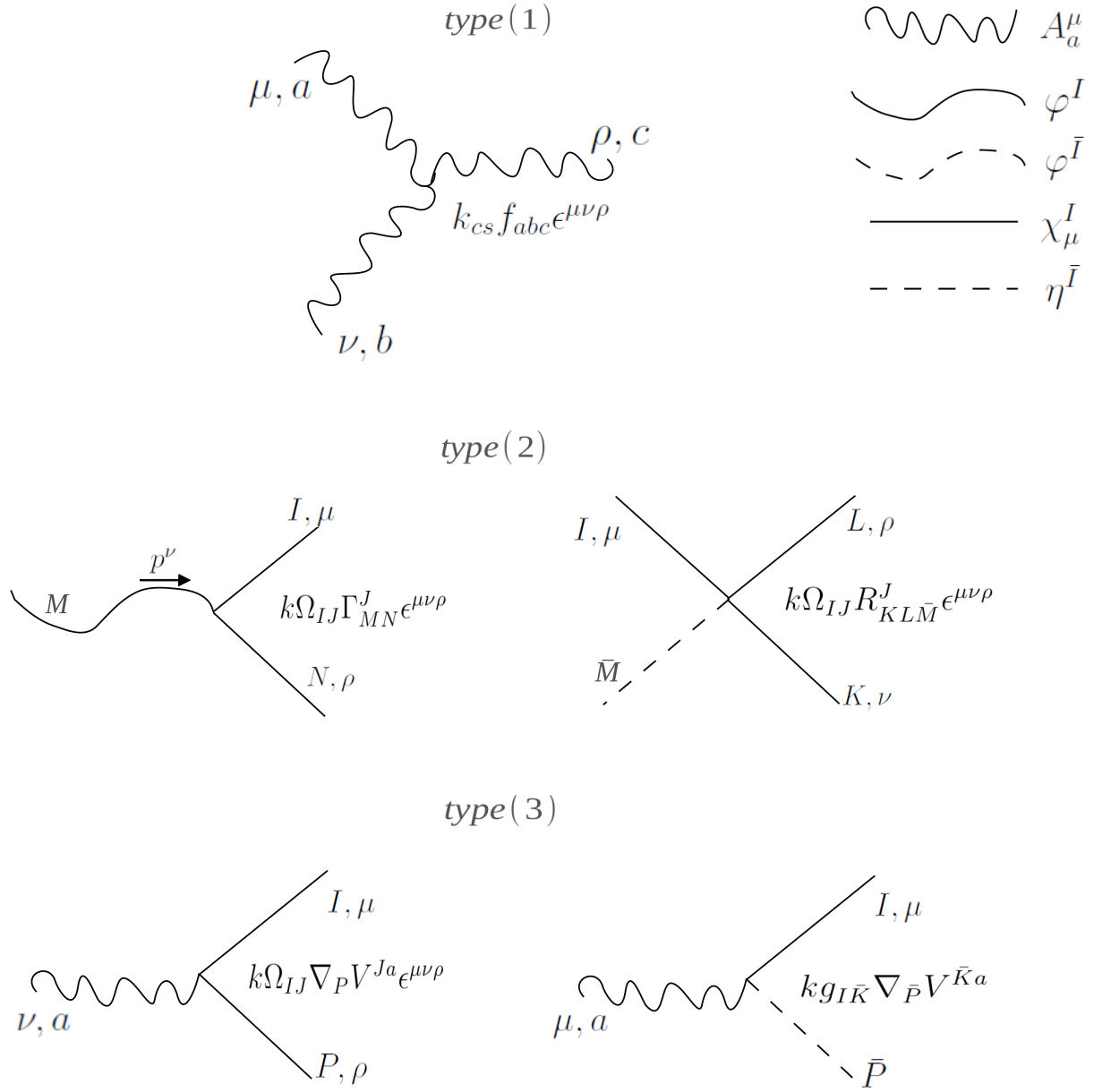
Figure 1: The Three Types of Vertices

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where L_{boson} is given in (3.13). To first order, the 3×3 matrix $\triangle_{\text{boson}}$ can be written as

$$\triangle_{\text{boson}} \sim \begin{pmatrix} 0 & \triangle^{\varphi\varphi} & \triangle^{\varphi A} \\ \triangle^{\varphi\varphi} & 0 & \triangle^{\varphi A} \\ \triangle^{A\varphi} & \triangle^{A\varphi} & \triangle^{AA} \end{pmatrix}, \tag{3.45}$$

where its components are spanned by all possible boson propagators:

$$\triangle^{(\varphi\varphi)I\bar{J}}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(\varphi\varphi)I\bar{J}}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(\varphi\varphi)}(M; A_0^\vartheta),$$

$$\triangle^{(AA)ab}_{\mu\nu}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k_{cs}} f^{(AA)ab}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(AA)}_{\mu\nu}(M; A_0^\vartheta), \tag{3.46}$$

$$\triangle^{(\varphi A)I,a}_{\mu}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(\varphi A)I,a}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(\varphi A)}_{\mu}(M; A_0^\vartheta).$$

Here, the labels ϕ_0 and A_0^ϑ mean that the corresponding quantities are evaluated at these values of the covariantly constant map ϕ_0 and flat connection A_0^ϑ. Notice that we can write the propagators as a product of two parts. The first part is a function $f(X, G; \phi_0, A_0^\vartheta)$ on the target manifold X that is characterized by the structural information of X and G. The second part is a function $\triangle'(M; A_0^\vartheta)$ on M.

Similarly, the propagator of the fermionic fields $\triangle_{\text{fermion}}$ can be obtained by solving the equation

$$k L_{\text{fermion}(\bar{I}\bar{K}, IK, ad, \mu\rho)}(x) \times \triangle^{(\bar{K}\bar{J}, KJ, db, \rho\nu)}_{\text{fermion}}(x - y) = \begin{pmatrix} \delta^{\bar{I}}_{\bar{J}} & 0 & 0 & 0 \\ 0 & \delta^I_J \delta^\mu_\nu & 0 & 0 \\ 0 & 0 & \delta^a_b & 0 \\ 0 & 0 & 0 & \delta^a_b \end{pmatrix} \cdot \delta(x - y), \tag{3.47}$$

where L_{fermion} is given in (3.15). To first order, the 4×4 matrix $\triangle_{\text{fermion}}$ can be written as

$$\triangle_{\text{fermion}} \sim \begin{pmatrix} 0 & \triangle^{\eta\chi} & 0 & 0 \\ \triangle^{\chi\eta} & \triangle^{\chi\chi} & \triangle^{\chi\bar{c}} & 0 \\ 0 & \triangle^{\bar{c}\chi} & 0 & \triangle^{c\bar{c}} \\ 0 & 0 & \triangle^{\bar{c}c} & 0 \end{pmatrix}. \tag{3.48}$$

where its components are spanned by all possible fermion propagators:

$$\triangle^{(\eta\chi)\bar{I}J}_{\mu}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(\eta\chi)\bar{I}J}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(\eta\chi)}_{\mu}(M; A_0^\vartheta),$$

$$\triangle^{(\chi\chi)IJ}_{\mu\nu}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(\chi\chi)IJ}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(\chi\chi)}_{\mu\nu}(M; A_0^\vartheta),$$

$$\triangle^{(\chi\bar{c})I,a}_{\mu}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(\chi\bar{c})I,a}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(\chi\bar{c})}_{\mu}(M; A_0^\vartheta), \tag{3.49}$$

$$\triangle^{(c\bar{c})ab}(X, G, M; \phi_0, A_0^\vartheta) = \frac{1}{k} f^{(c\bar{c})ab}(X, G; \phi_0, A_0^\vartheta) \triangle^{'(c\bar{c})}(M; A_0^\vartheta).$$

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Similar to the boson propagators, we can also write these fermion propagators as the product of two parts.

An Equivariant Linking Number of Knots

Notice here that we may regard $\Delta'^{(\chi\chi)}_{\mu\nu}(M; A_0^\vartheta)$ as an equivariant one-form depending on A_0^ϑ. This means that for one-cycles C' in M which satisfy the following equivariant Stoke's theorem

$$\int_{C'} d_{A_0^\vartheta} \mathcal{F} = \int_{\partial_{A_0^\vartheta} C'} \mathcal{F} = 0, \tag{3.50}$$

the double integral

$$\oint_{C'_1} dx^\mu \oint_{C'_2} dx^\nu \Delta'^{(\chi\chi)}_{\mu\nu}(M; A_0^\vartheta) \tag{3.51}$$

would define an "*equivariant linking number*" of knots C'_1 and C'_2.

3.6. New Three-Manifold Invariants and Weight Systems

We would now like to show that by computing the perturbative partition function, we would be able to derive *new* three-manifold invariants and their associated weight systems which depend on both G and X. To this end, let us first review the three-manifold invariants and their associated weight systems that come from Chern-Simons and Rozansky-Witten theory.

Three-Manifold Invariants and Weight Systems From Chern-Simons Theory

The perturbative partition function of Chern-Simons theory can be written as

$$Z_{CS}(M; G; k_{cs}) = \sum_m Z^{(m)}_{CS}(M; G; k_{cs}), \tag{3.52}$$

where (m) denotes the order of k_{cs} in the indicated term. If the classical solution A_0 is the trivial flat connection over M, the propagators would be independent of A_0. Then, the partition function would take (up to a one-loop contribution) the very simple form

$$Z^{(\mathrm{tr})}_{CS}(M; G; k_{cs}) = \exp\left(\sum_{m=1}^\infty S_{G,m+1}(M)\, k_{cs}^{-m}\right), \tag{3.53}$$

where

$$S_{G,m+1} = \sum_{\Gamma \in \Gamma_{3,m+1}} a_\Gamma(G) I_\Gamma(M). \tag{3.54}$$

Here, the sum runs over all trivalent Feynman graphs $\Gamma_{3,m+1}$ with $m+1$ loops (and $2m$ vertices),[4] and $I_\Gamma(M)$ are the integrals over $M \times M \times \cdots \times M$ of the products of propagators.

The Jacobi identity of the Lie algebra of G is used to show that although the individual integrals $I_\Gamma(M)$ depend on the metric of M, the metric-dependence cancels out of the sum in (3.54) [4, 5]. Thus, $S_{G,m+1}$ and therefore $Z_{CS}(M; G; k_{cs})$, are indeed topological invariants of the three-manifold M. Furthermore, because the factor $a_\Gamma(G)$ can be regarded as a weight factor weighting each graph term, $S_{G,m+1}$ also defines what is called a weight system. Clearly, this weight system depends on Lie algebra structure.

Three-Manifold Invariants and Weight Systems From Rozansky-Witten Theory

The perturbative partition function of Rozansky-Witten theory can (up to a one-loop contribution) be written as

$$Z(M, X) = \sum_\Gamma Z_\Gamma(M, X), \tag{3.55}$$

where $\sum_\Gamma$ is a summation over all relevant Feynman graphs of the theory, and

$$Z_\Gamma(M, X) = b_\Gamma(X) \sum_b I_{\Gamma,b}(M), \tag{3.56}$$

where $\sum_b$ denotes the summation of all possible ways of assigning the vertices to each Feynman graph. Here, $I_{\Gamma,b}$ are the integrals over $M \times M \times \cdots \times M$ of the products of propagators as well as of the relevant one-form fermionic zero modes. $I_{\Gamma,b}$ just depends on the structure of M, while b_Γ serves as a weight factor which depends on the curvature tensor of the target space X that comes from the underlying vertices. Thus, $Z_\Gamma(M, X)$ defines a weight system. Clearly, this weight system depends on hyperkähler geometry.

The Bianchi identity plays the same role here as the Jacobi identity in CS theory [7]; one can use it to show that the dependence on the metric of M cancels out of the sum (3.55), i.e., $Z(M, X)$ is a topological invariant of the three-manifold M.

Coming Back to Our Theory

Coming back to our theory, we can, after evaluating the path integral, write the perturbative partition function as

$$Z(M, X, G) = \sum_{A_0^\vartheta} e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)} \cdot Z_0(A_0^\vartheta) \cdot Z(M, X, G; A_0^\vartheta; k_{cs}), \tag{3.57}$$

where $e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)}$ is the topological factor coming from the Chern-Simons part of the total Lagrangian evaluated at a flat connection A_0^ϑ; $Z_0(A_0^\vartheta)$ is the topological one-loop contribution

---

[4]For a description of a (trivalent) Feynman graph, see [7].

given in (3.28); and

$$Z(M, X, G; A_0^\vartheta; k_{cs}) = \sum_\Gamma Z_\Gamma(M, X, G; A_0^\vartheta; k_{cs}^m), \qquad (3.58)$$

where $\sum_\Gamma$ is a sum over all possible Feynman diagrams with two or more loops that (i) have the right number of fermionic zero modes to absorb those that appear in the path integral measure, and (ii) are free of the coupling constant k. Here, the label k_{cs}^m (where m may vanish) means that Γ carries with it a factor of k_{cs}^m.

In fact, Z_Γ can be expressed as

$$Z_\Gamma(M, X, G; A_0^\vartheta; k_{cs}^m) = \int_{\mathcal{M}^\vartheta} \sqrt{g} \, d^{2n}\phi_0^I \, d^{2n}\phi_0^{\bar{I}} \, W_\Gamma(X, G; \phi_0, A_0^\varphi) I_\Gamma(M, X, G; \phi_0, A_0^\vartheta; k_{cs}^m), \tag{3.59}$$

where I_Γ is an integral over $M \times M \times \cdots \times M$ of the products of propagators as well as of the one-form fermionic zero modes $\omega_\mu(x)$ in (3.32), while the weight factor W_Γ is a product of terms relevant to Γ that are associated with the vertices in Fig. 1.

We can characterize the partition function by classifying the Feynman diagrams into three categories as follows.

(1) <u>Chern-Simons-Type Diagrams</u>. These diagrams result purely from the vertices $A \wedge A \wedge A$. Thus, they correspond to diagrams in usual Chern-Simons theory. The topological property of Chern-Simons-type diagrams has already been verified in earlier works [4, 5]. As such, we would have nothing more to add about them.

To discuss the next two types of diagrams, we take, for simplicity, the case where $b_1' = 0$ and $b_0' = 1$. Then, the nonvanishing Feynman diagrams must contain exactly $2nb_0'$ zero modes η_0^I. For brevity, we will only discuss diagrams whose vertices emanate 4 legs.

(2) <u>Diagrams Free of Gauge Fields</u>. Since these diagrams result from vertices which are free of the gauge field A, they do not contain the gauge field propagator. Examples of such diagrams are given by the A pattern and B pattern in Fig. 2.
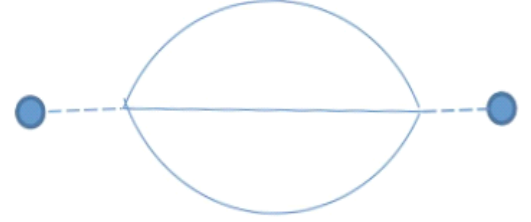
The A pattern diagram is formed by the vertex

$$k\partial_K \Gamma_{I\bar{M}\bar{N}} \partial_\mu \varphi^{\bar{M}} \varphi^K \chi^{I,\mu} \eta_0^{\bar{N}},$$
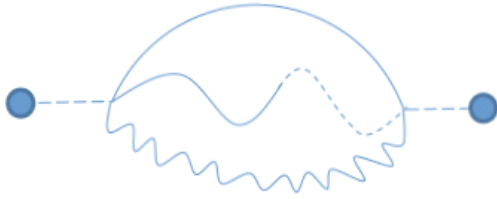
and the propagators in the diagram are $\triangle^{(\chi\chi)}$ and $\triangle^{(\varphi\varphi)}$. Therefore for the A pattern
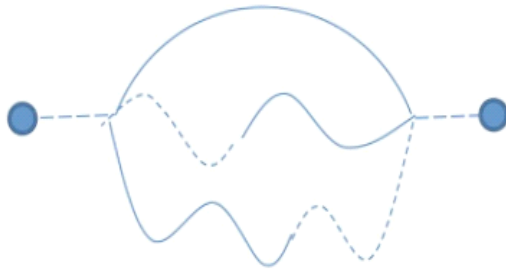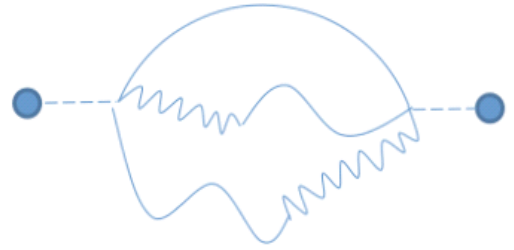
A pattern

B pattern

C pattern

D pattern

A_a^μ

φ^I

$\varphi^{\bar{I}}$

χ_μ^I

$\eta^{\bar{I}}$
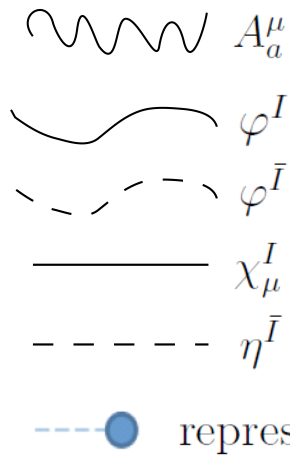
represents zero mode $\eta_0^{\bar{I}}$

Figure 2: A, B, C and D Pattern Diagrams

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diagram, the terms in (3.59) are

$$I_\Gamma(M, X, G; \phi_0, A_0^\vartheta) = \int_M \prod_{l=1}^n (\triangle_{\mu\nu}^{(\chi\chi)I_l J_l} \partial^\mu \triangle^{(\varphi\varphi)L_l \bar{P}_l} \partial^\nu \triangle^{(\varphi\varphi)K_l \bar{Q}_l})(x_l, y_l) \, d^3 x_l d^3 y_l \qquad (3.60)$$

and

$$W_\Gamma(X, G; \phi_0^I) = \epsilon^{\bar{M}_1 \dots \bar{M}_{2n}} \prod_{l=1}^n (\partial_{K_l} \partial_{\bar{M}_{1,l}} g_{\bar{P}_l I_l})(\partial_{L_l} \partial_{\bar{M}_{2,l}} g_{\bar{Q}_l J_l}) \qquad (3.61)$$

where the $\epsilon^{\bar{M}_1 \dots \bar{M}_{2n}}$ factor comes from the expectation value of the zero modes η_0 defined in (3.31). The contribution of this diagram to the partition function can then be evaluated by substituting the above two expressions in (3.59).

The B pattern diagram is formed by the vertex

$$k\Omega_{IJ} R^J_{KL\bar{M}} \epsilon^{\mu\nu\rho} \chi_\mu^I \chi_\nu^K \chi_\rho^L \eta_0^{\bar{M}},$$

and the propagator in the diagram is $\triangle^{(\chi\chi)}$. Therefore for the B pattern diagram, the terms in (3.59) are

$$I_\Gamma(M, G, X; \phi_0^I, A_0^\vartheta) = \int_M \prod_{l=1}^n (\epsilon^{\mu_1 \nu_1 \rho_1} \epsilon^{\mu_2 \nu_2 \rho_2} \triangle_{\mu_1 \mu_2}^{(\chi\chi)I_{1,l} I_{2,l}} \triangle_{\nu_1 \nu_2}^{(\chi\chi)K_{1,l} K_{2,l}} \triangle_{\rho_1 \rho_2}^{(\chi\chi)L_{1,l} L_{2,l}})(x_l, y_l) \, d^3 x_l d^3 y_l$$

$$(3.62)$$

and

$$W_\Gamma(X, G; \varphi_0^I) = \epsilon^{\bar{M}_1 \dots \bar{M}_{2n}} \prod_{l=1}^n (\Omega_{I_{1,l} J_{1,l}} R^{J_{1,l}}_{K_{1,l} L_{1,l} \bar{M}_{1,l}})(\Omega_{I_{2,l} J_{2,l}} R^{J_{2,l}}_{K_{2,l} L_{2,l} \bar{M}_{2,l}}) \qquad (3.63)$$

Notice that the A and B pattern diagrams in Fig. 2 are similar to those in RW theory. Nevertheless, unlike RW theory, our propagator factor I_Γ depends on the flat gauge field A_0^ϑ. If A_0^ϑ were trivial, the contributions of the A and B pattern diagrams to the partition function would be as given in RW theory, as expected.

(3) <u>Diagrams Mixing Gauge and Matter Fields</u>. Examples of such diagrams are given by the C pattern and D pattern in Fig. 2.

The C pattern diagram is formed by the vertices

$$k\partial_K(\nabla_{\bar{M}} V_I^a)\eta_0^{\bar{M}} \chi_\mu^I A_a^\mu \varphi^K \quad \text{and} \quad k\partial_{\bar{K}}(\nabla_{\bar{M}} V_I^a)\eta_0^{\bar{M}} \chi_\mu^I A_a^\mu \varphi^{\bar{K}},$$

and the propagators in the diagram are $\triangle^{(\chi\chi)}$, $\triangle^{(\varphi\varphi)}$, $\triangle^{(AA)}$. Therefore for the C pattern diagram, the terms in (3.59) are

$$I_\Gamma(M, X, G; \phi_0^I, A_0^\vartheta; k_{cs}^{-n}) = \int_M \prod_{l=1}^n (\triangle_{\mu\nu}^{(\chi\chi)I_l J_l} \triangle^{(\varphi\varphi)K_l \bar{L}_l} \triangle_{a_l b_l}^{(AA)\mu\nu})(x_l, y_l) \, d^3 x_l d^3 y_l \qquad (3.64)$$

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and

$$W_\Gamma(X, G; \varphi_0^I) = \epsilon^{\bar{M}_1 \ldots \bar{M}_{2n}} \prod_{l=1}^{n} \partial_{\bar{L}_l}(\nabla_{\bar{M}_{1,l}} V_{I_l}^{a_l}) \partial_{K_l}(\nabla_{\bar{M}_{2,l}} V_{J_l}^{b_l}) \tag{3.65}$$

The D pattern diagram is formed by the vertex

$$k \partial_K (\nabla_{\bar{M}} V_I^a) \eta_0^{\bar{M}} \chi_\mu^I A_a^\mu \varphi^K,$$

and the propagators in the diagram are $\triangle^{(\chi\chi)}$, $\triangle^{(\varphi A)}$. Therefore for the D pattern diagram, the terms in (3.59) are

$$I_\Gamma(M, X, G; \phi_0^I, A_0^\vartheta) = \int_M \prod_{l=1}^{n} (\triangle_{\mu a_l}^{(\varphi A)K_{1,l}} \triangle_{\nu b_l}^{(\varphi A)K_{2,l}} \triangle^{(\chi\chi)I_{1,l} I_{2,l}, \mu\nu})(x_l, y_l) \, d^3 x_l d^3 y_l \tag{3.66}$$

and

$$W_\Gamma(X, G; \varphi_0^I) = \epsilon^{\bar{M}_1 \ldots \bar{M}_{2n}} \prod_{l=1}^{n} \partial_{K_{1,l}}(\nabla_{\bar{M}_{1,l}} V_{I_{1,l}}^{a_l}) \partial_{K_{2,l}}(\nabla_{\bar{M}_{2,l}} V_{I_{2,l}}^{b_l}) \tag{3.67}$$

New Three-Manifold Invariants and Weight Systems

As shown in (3.46) and (3.49), the propagators $\triangle$ can be expressed as the product of a function $f(X, G)$ on the target manifold X and a function $\triangle'(M)$ on the three-manifold M. Therefore, we can rewrite the above propagator factors as

$$I_\Gamma(M, X, G; \phi_0, A_0^\vartheta; k_{cs}^m) = f_\Gamma(X, G; \phi_0, A_0^\vartheta) I_\Gamma'(M; A_0^\vartheta; k_{cs}^m) \tag{3.68}$$

where I_Γ' is a function on M that depends on the flat gauge field A_0^ϑ and which carries a factor of k_{cs}^m, and the function f_Γ is characterized, among other things, by the structure of the target space X and the gauge group G. In turn, this means that we can rewrite (3.59) as

$$Z_\Gamma(M, X, G; A_0^\vartheta; k_{cs}^m) = \mathscr{W}_\Gamma(X, G; A_0^\vartheta) \, I_\Gamma'(M; A_0^\vartheta; k_{cs}^m) \tag{3.69}$$

where

$$\mathscr{W}_\Gamma(X, G; A_0^\vartheta) = \int_{\mathcal{M}^\vartheta} \sqrt{g} \, d^{2n} \phi_0^I \, d^{2n} \phi_0^{\bar{I}} \, W_\Gamma(X, G; \phi_0, A_0^\vartheta) f_\Gamma(X, G; \phi_0, A_0^\vartheta) \tag{3.70}$$

can be regarded as a weight factor which combines the structural information of the hyperkähler manifold X and the Lie algebra $\mathfrak{g}$ of the gauge group G.

As in CS and RW theory, I_Γ' in (3.69) can be expected to depend on the metric of M. However, since the partition function in (3.57) and therefore

$$Z(M, X, G; A_0^\vartheta; k_{cs}) = \sum_\Gamma \mathscr{W}_\Gamma(X, G; A_0^\vartheta)\, I_\Gamma'(M; A_0^\vartheta; k_{cs}^m) \tag{3.71}$$

are topological on M at the outset, the metric-dependence of I_Γ' should cancel out in the sum (3.71). One should be able to rigorously show this cancellation by using the Jacobi identity of G, the Bianchi identity of X, and the geometric identities of the moment maps discussed in section 2.1.

In summary, our perturbative partition function furnishes us with a *new* three-manifold invariant $Z(M, X, G; A_0^\vartheta; k_{cs})$ which depends on both G and X, that also defines a *new* weight system whose weights $\mathscr{W}_\Gamma(X, G; A_0^\vartheta)$ are characterized by both Lie algebra structure *and* hyperkähler geometry.

4. Canonical Quantization and the Nonperturbative Partition Function

4.1. Canonical Quantization

Let us now canonically quantize our gauged sigma model on M with target space X. To this end, first note that the Hamiltonian $H = \langle \int T_{00} \rangle = \langle \delta_{\widehat{Q}} \mathcal{O} \rangle = 0$. Next, note that locally, the Riemannian manifold M can be written as $\Sigma \times I$, where I is the 'time' dimension and Σ is a compact Riemann surface. Thus, since $H = 0$ whence the theory should be time-independent, it would mean that we can just analyze the physics over any $\Sigma \times I \subset M$.

This property of $H = 0$ also means that only ground states contribute to the spectrum of the theory. Therefore, where the fermions are concerned, only the zero modes contribute to the physical Hilbert space. Where the gauge field is concerned, only the classical configuration of flat connections A_0 contribute to the physical Hilbert space. And where the bosons are concerned, only the covariantly constant maps ϕ from M to X which satisfy $D_\mu \phi = \partial_\mu \phi + A_{0\mu}^a V_a = 0$, contribute to the physical Hilbert space.

Let τ be the time coordinate. Then, according to the last paragraph, ϕ would satisfy $\partial_\tau \phi + A_{0\tau}^a V_a = 0$. In the gauge where $A_\tau = 0$, we would also have

$$\partial_\tau \eta^{\bar{I}} = 0, \quad \partial_\tau \chi_\tau^I = 0, \quad \partial_\tau \chi_\mu^I = 0, \tag{4.1}$$

where $\eta^{\bar{I}}$, χ_τ^I and χ_μ^I are fermionic zero modes. In other words, the zero modes ϕ, η^I, χ_τ^I

and χ_μ^I are τ-independent, which means that we effectively have a two-dimensional gauged sigma model on Σ.

The Commutation and Anticommutation Relations

From the Lagrangian, we compute the momentum conjugate of η, χ and A to be

$$\frac{\delta L}{\delta \partial^\tau \eta^{\bar{K}}} = g_{I\bar{K}}\chi_\tau^I,$$

$$\frac{\delta L}{\delta \partial_\tau \chi_\mu^I} = \epsilon^{\tau\mu\nu}\Omega_{IJ}\chi_\nu^J,$$

$$\frac{\delta L}{\delta \partial^\tau \chi_\tau^I} = g_{\bar{K}I}\eta^{\bar{K}}, \tag{4.2}$$

$$\frac{\delta L}{\delta \partial_\tau A_{0\mu}^a} = \epsilon^{\tau\mu\nu}k_{ab}A_{0\nu}^b.$$

Note at this point from (3.15) that the fermionic zero modes χ_μ^I are solutions of the covariant equation

$$\Omega_{IJ}\epsilon^{\tau\mu\nu}\partial_\mu\chi_\nu^J + \Omega_{IK}\epsilon^{\tau\mu\nu}\partial_J V_a^K A_{0\mu}^a\chi_\nu^J = 0; \quad \mu, \nu = 1, 2, \tag{4.3}$$

which depend on the choice of the flat connection A_0; in other words, we can write $\chi_\mu^I(x) = \chi_\alpha^I \omega_\mu^\alpha(A_0, x)$, where ω^α are covariant harmonic one-forms on Σ, and χ_α^I are constant fermionic coefficients. Hence, if $\int_\Sigma \omega^\alpha \wedge \omega^\beta = L^{\alpha,\beta}$, the relations in (4.2) tell us that the commutation and anticommutation relations upon quantizing the zero modes must be

$$\{\eta^{\bar{I}}, \chi_\tau^J\} = \frac{1}{k}g^{\bar{I}J},$$

$$\{\chi_\alpha^I, \chi_\beta^J\} = \frac{1}{k}\Omega^{IJ}(L^{-1})_{\alpha,\beta}, \tag{4.4}$$

$$\left[A_{0\mu}^a(x)A_{0\nu}^b(y)\right] = \frac{1}{k_{cs}}\epsilon_{\mu\nu}\delta^{ab}\delta^2(x-y),$$

where g and Ω are evaluated at the covariantly constant map ϕ.

A Relevant Digression

Before proceeding any further, let us discuss the following important point. Recall from section 3 that after gauge-fixing, it is the $\widehat{Q}$-cohomology that is relevant. Nevertheless, the spectrum of the theory is unchanged by gauge-fixing, and so the Q- and $\widehat{Q}$-cohomology ought to be equivalent. Let us now verify this claim.

First, recall that we have

$$Q^2 = \text{gauge transformation}, \tag{4.5}$$

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and

$$\widehat{Q} = Q + Q_{\text{FP}}, \quad \text{where} \quad \widehat{Q}^2 = 0. \tag{4.6}$$

Second, by definition, we have

$$\ker(\widehat{Q}) = \{\mathcal{O}|\{Q + Q_{\text{FP}}, \mathcal{O}] = 0\}. \tag{4.7}$$

From (2.11), we find that $\{Q, \mathcal{O}] \neq -\{Q_{\text{FP}}, \mathcal{O}]$. So,

$$\ker(\widehat{Q}) = \ker(Q) \cap \ker(Q_{\text{FP}}). \tag{4.8}$$

That is, $\{Q, \mathcal{O}] = 0$ and $\{Q_{\text{FP}}, \mathcal{O}] = 0$, which means

$$\{\{Q, Q_{\text{FP}}\}, \mathcal{O}] = 0. \tag{4.9}$$

Third, from the definition of Q_{FP}, we have

$$\{Q_{\text{FP}}^2, \mathcal{O}] = 0. \tag{4.10}$$

Also, we have

$$\{\widehat{Q}^2, \mathcal{O}] = \{Q^2 + \{Q, Q_{\text{FP}}\} + Q_{\text{FP}}^2, \mathcal{O}] = 0. \tag{4.11}$$

Thus, from (4.9), (4.10) and (4.11), we have

$$\{Q^2, \mathcal{O}] = 0. \tag{4.12}$$

In turn, this means that

$$\ker(\widehat{Q}) = \ker(Q) \cap \ker(Q_{\text{FP}}) = \ker(Q) \cap \{\mathcal{O}|\{Q^2, \mathcal{O}] = 0\}. \tag{4.13}$$

Last, note that in

$$\text{im}(\widehat{Q}) = \text{im}(Q + Q_{\text{FP}}), \tag{4.14}$$

because $\text{im}(Q_{\text{FP}})$ contains ghost fields, $\text{im}(Q_{\text{FP}})$ does not contribute to the physical Hilbert space. Hence,

$$\text{im}(\widehat{Q}) \cong \text{im}(Q). \tag{4.15}$$

Altogether, this means that

$$\mathcal{O} \in \frac{\ker(\widehat{Q})}{\text{im}(\widehat{Q})} = \frac{\ker(Q) \cap \ker(Q_{\text{FP}})}{\text{im}(Q + Q_{\text{FP}})} \cong \frac{\ker(Q) \cap \{\mathcal{O}|\{Q^2, \mathcal{O}] = 0\}}{\text{im}(Q)}, \tag{4.16}$$

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which verifies our claim that the $\widehat{Q}$- and Q-cohomology are equivalent. Therefore, let us henceforth focus on the Q-cohomology; in particular, let us proceed to ascertain the relevant Hilbert space of states in the Q-cohomology.

The Hilbert Space of States

To this end, note that since we are restricting ourselves to the classical configuration A_0 that is free of interacting fluctuations, we can view the total theory as a CS theory plus a non-dynamically gauged RW theory. As such, any state $|\Psi\rangle$ in the Q-cohomology ought to take the form

$$\boxed{|\Psi\rangle = |\psi\rangle \otimes \tilde{\Phi}|0\rangle} \tag{4.17}$$

Here, $|\psi\rangle$ is a state in the CS theory which is associated with a Q-closed but not Q-exact wave function $\psi(A_0^\vartheta)$ that depends on a flat gauge field A_0^ϑ along Σ, where [11]

$$\psi(A_0^\vartheta) = \int_{A|_\Sigma = A_0^\vartheta} DA\, e^{-S_{cs}}, \tag{4.18}$$

and $\tilde{\Phi}$ is a Q-closed but not Q-exact state operator of the non-dynamically gauged RW theory. Let us now determine $\tilde{\Phi}$.

From (4.4), it is clear that the vacuum state $|0\rangle$ would be annihilated by the operators χ_β^I and χ_τ^I.[5] Hence, a first-cut construction of an arbitrary state $|\Phi\rangle$ of the non-dynamically gauged RW theory would be

$$|\Phi\rangle = \tilde{\Phi}|0\rangle = \Phi_{I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_k}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l}\eta^{\bar{I}_1} \cdots \eta^{\bar{I}_k}|0\rangle. \tag{4.19}$$

Generically, Φ depends on the covariantly constant map ϕ; hence, a natural generically-nonvanishing scalar product of states would be given by

$$\langle\Phi^{(1)}|\Phi^{(2)}\rangle = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi^I d^{2n}\phi^{\bar{I}}\, \epsilon^{I_1 \ldots I_{2n}\bar{I}_1 \ldots \bar{I}_{2n}}\, \Phi_{I_1 \cdots I_q \bar{I}_1 \cdots \bar{I}_p}^{(1)}\, \Phi_{I_{q+1} \cdots I_{2n}\bar{I}_{p+1} \cdots \bar{I}_{2n}}^{(2)} = \int_{\mathcal{M}^\vartheta} \Phi^{q+p} \wedge \Phi^{4n-q-p}, \tag{4.20}$$

where Φ^m is an m-form on $\mathcal{M}^\vartheta$, the space of all physically distinct ϕ's for some A_0^ϑ. In other words, $\tilde{\Phi}$ would correspond to an element of $\Omega^{l+k}(\mathcal{M}^\vartheta)$, the space of all $(l+k)$-forms on $\mathcal{M}^\vartheta$.

Now, from (2.8), the fields transform under the supercharge Q as

$$\delta_Q\eta^{\bar{I}_i} = -V_a^{\bar{I}_i}\mu_+^a,$$
$$\delta_Q\phi^{\bar{I}} = \eta^{\bar{I}}, \tag{4.21}$$
$$\delta_Q\chi^I = D\phi^I.$$

---

[5] For ease of illustration, we henceforth assume that $b_0' = 1$ and $b_1' = 1$.

At the level of zero modes, the last equation $\delta_Q \chi^I = D\phi^I = 0$. If $\tilde{\Phi}$ is d-closed, i.e., $\partial_{\bar{J}} \Phi(\phi) = \partial_J \Phi(\phi) = 0$, we would have

$$
\begin{aligned}
\delta_Q \tilde{\Phi} &= \sum_i (-1)^{l+i} \Phi_{I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_k} V_a^{\bar{I}_i} \mu_+^a \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \widehat{\eta}^{\bar{I}_i} \cdots \eta^{\bar{I}_k} \\
&= (-1)^l k \, \Phi_{I_1 \cdots I_l \bar{I}_1 \bar{I}_2 \cdots \bar{I}_k} V_a^{\bar{I}_1} \mu_+^a \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_2} \cdots \eta^{\bar{I}_k} \\
&\neq 0.
\end{aligned}
\tag{4.22}
$$

Thus, the state operator $\tilde{\Phi}$ is not Q-closed, as we would like it to be.

We can try to 'improve' it to

$$
\tilde{\Phi} = \Phi_{I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_k}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_k} - \mu_+^a \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-2}}, \tag{4.23}
$$

where now,

$$
\begin{aligned}
\delta_Q \tilde{\Phi} = {} & (-1)^l \left[k V_a^{\bar{I}_1} \Phi_{I_1 \cdots I_l \bar{I}_1 \bar{I}_2 \cdots \bar{I}_k} \right] \mu_+^a \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_2} \cdots \eta^{\bar{I}_k} \\
& - (-1)^l \left[\partial_{\bar{K}} \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}} \right] \mu_+^a \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{K}} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-2}} \\
& - (-1)^l (k-2) \mu_+^a \mu_+^b V_b^{\bar{I}_1} \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}} \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_2} \cdots \eta^{\bar{I}_{k-2}},
\end{aligned}
\tag{4.24}
$$

after exploiting the fact that μ_+ is holomorphic. If moreover, $\Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}}$ is anti-holomorphic and

$$
i_a(\Phi) = d\Phi_a, \tag{4.25}
$$

where $(i_a(\Phi))_{\bar{I}_2 \cdots I_l \bar{I}_1 \bar{I}_2 \cdots \bar{I}_k} = k V_a^{\bar{I}_1} \Phi_{I_1 \cdots I_l \bar{I}_1 \bar{I}_2 \cdots \bar{I}_k}$ is a contraction with V_a of $\Phi \in \Omega^{l+k}(\mathcal{M}^\vartheta)$, and $(d\Phi_a)_{\bar{K} I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}} = \partial_{\bar{K}} \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}}$ is an exterior derivative of $\Phi_a \in \mathfrak{g}^* \otimes \Omega^{l+k-2}(\mathcal{M}^\vartheta)$, the second term on the RHS of (4.24) would simply cancel the first one out, i.e., we would have

$$
\delta_Q \tilde{\Phi} = (-1)^{l+1} (k-2) \mu_+^a \mu_+^b V_b^{\bar{I}_1} \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}} \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_2} \cdots \eta^{\bar{I}_{k-2}}. \tag{4.26}
$$

Hence, by 'improving' $\tilde{\Phi}$ via (4.23), we have actually made progress: by comparing (4.26) and (4.22), it is clear that we have gone from having k to $k-2$ many η fields in the expression for $\delta_Q \tilde{\Phi}$.

We can continue to 'improve' $\tilde{\Phi}$ by adding more terms of lower order in η:

$$
\begin{aligned}
\tilde{\Phi} = {} & \Phi_{I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_k}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_k} - \mu_+^a \Phi_{a I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-2}}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-2}} \\
& + \mu_+^a \mu_+^b \Phi_{ab I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-4}}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-4}} \\
& - \mu_+^a \mu_+^b \mu_+^c \Phi_{abc I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_{k-6}}(\phi)\chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-6}} + \ldots.
\end{aligned}
\tag{4.27}
$$

Here, even-valued k is such that $0 < k \leq \dim_{\mathbb{C}}(\mathcal{M}^{\vartheta})$, and $\Phi_a, \Phi_{ab}, \Phi_{abc}, \cdots \in S(\mathfrak{g}^*) \otimes \Omega(\mathcal{M}^{\vartheta})$ are anti-holomorphic, where $S(\mathfrak{g}^*)$ is the symmetric algebra on $\mathfrak{g}^*$. If moreover,

$$i_a(\Phi) = d\Phi_a, \quad i_b(\Phi_a) = d\Phi_{ab}, \quad i_c(\Phi_{ab}) = d\Phi_{abc}, \ldots, \tag{4.28}$$

one will find that $\delta_Q \tilde{\Phi} = 0$.

From the field variations in (4.21) and the comment thereafter, one can see that Q effectively acts on $\tilde{\Phi}$ as $d - \mu_+^a i_a$. Together with (4.27) and (4.28), it would mean that for $\tilde{\Phi}$ to be Q-closed but not Q-exact, it must correspond to a class in the G-equivariant cohomology $H_G(\mathcal{M}^{\vartheta})$.

It is now clear from (4.17) and the fact that $\tilde{\Phi}$ corresponds to a class in $H_G(\mathcal{M}^{\vartheta})$, that the relevant Hilbert space $\mathcal{H}$ of all states $|\Psi\rangle$ in the Q-cohomology can be expressed as

$$\boxed{\mathcal{H} = \mathcal{H}_{\mathrm{CS}}(A_0^{\vartheta}, \Sigma) \otimes H_G(\mathcal{M}^{\vartheta})} \tag{4.29}$$

where $\mathcal{H}_{\mathrm{CS}}(A_0^{\vartheta}, \Sigma)$ is the Hilbert space of states in CS theory associated with wave functions $\psi(A_0^{\vartheta})$ in the Q-cohomology that depend on a flat gauge field A_0^{ϑ} on Σ.

An Example

Before we end this subsection, let us consider the case where $\Sigma = \mathbf{S}^2$, G is some arbitrary compact simple Lie group, $X = T^*(G/\mathbb{T})$, and $\mathbb{T} \subset G$ is a maximal torus. For simply-connected $\Sigma = \mathbf{S}^2$, we can go to pure gauge on Σ whence we can regard the flat gauge field A_0^{ϑ} to be trivial in all directions (since $A_{0\tau}^{\vartheta} = 0$ also). Consequently, $\mathcal{H}_{\mathrm{CS}}$ is trivial, χ_{μ} and η would become ordinary harmonic forms on Σ, and the ϕ's would just be constant maps whence $\mathcal{M}^{\vartheta} = X = T^*(G/\mathbb{T})$. Therefore, the corresponding Hilbert space would simply be

$$\mathcal{H}_G = H_G(T^*(G/\mathbb{T})), \tag{4.30}$$

which is the G-equivariant cohomology of $T^*(G/\mathbb{T})$. In other words, via the Cartan model of equivariant cohomology, we have (c.f. [12])

$$\mathcal{H}_G \cong H\left([S(\mathfrak{g}^*) \otimes \Omega(T^*(G/\mathbb{T}))]_{G\text{-invariant}}\right). \tag{4.31}$$

Here, $H(\ldots)$ is the cohomology of the complex with Cartan differential $d_G = 1 \otimes d + F^a \otimes i_a$, where F^a is some $\mathfrak{g}$-valued function on $T^*(G/\mathbb{T})$ of degree two.

From our discussion leading up to (4.27), and the fact that $b_1(\mathbf{S}^2) = 0$ and $b_0(\mathbf{S}^2) = 1$ whence there are no χ_{μ}'s but $\dim_{\mathbb{C}}(T^*(G/\mathbb{T}))$ many η's, we find that a generic arbitrary state in $\mathcal{H}_G$ would be given by

$$|\Psi\rangle = \tilde{\Phi}|0\rangle, \tag{4.32}$$

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where

$$\tilde{\Phi} = \Phi_{\bar{I}_1 \cdots \bar{I}_k}(\phi)\eta^{\bar{I}_1} \cdots \eta^{\bar{I}_k} - \mu_+^a \Phi_{a\bar{I}_1 \cdots \bar{I}_{k-2}}(\phi)\eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-2}}$$
$$+ \mu_+^a \mu_+^b \Phi_{ab\bar{I}_1 \cdots \bar{I}_{k-4}}(\phi)\eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-4}} \tag{4.33}$$
$$- \mu_+^a \mu_+^b \mu_+^c \Phi_{abc\bar{I}_1 \cdots \bar{I}_{k-6}}(\phi)\eta^{\bar{I}_1} \cdots \eta^{\bar{I}_{k-6}} + \dots .$$

Here, even-valued k is such that $0 < k \leq \dim_{\mathbb{C}}(T^*(G/\mathbb{T}))$, and $\Phi_a, \Phi_{ab}, \Phi_{abc}, \cdots \in S(\mathfrak{g}^*) \otimes \Omega(T^*(G/\mathbb{T}))$ are anti-holomorphic.

That being said, it can be shown [13] that $\mathcal{H}_G = S(\mathfrak{t}^*)$, where $\mathfrak{t}$ is the Lie algebra of T. In other words, an arbitrary state in $\mathcal{H}_G$ ought to be given by

$$\boxed{|\Psi\rangle = (-1)^p \, \mu_+^{a_1} \mu_+^{a_2} \cdots \mu_+^{a_p} \Phi_{a_1 a_2 \dots a_p}(\phi)|0\rangle} \tag{4.34}$$

where $1 \leq a_i \leq \text{rank}(G)$, and p is any positive integer.

Take for example $G = SU(2)$ and $X = T^*(\mathbf{CP}^1)$, where $\text{rank}(G) = 1$ and $\dim_{\mathbb{C}}(X) = 2$. Then, the only state in $\mathcal{H}_{SU(2)}$ is

$$|\Psi^{(1)}\rangle = -\mu_+^1 \Phi_1(\phi)|0\rangle. \tag{4.35}$$

Take as another example $G = SU(N)$ and $X = T^*(SU(N)/U(1)^{N-1})$, where $\text{rank}(G) = N-1$ and $\dim_{\mathbb{C}}(X) = N(N-1)$. Then, the independent states in $\mathcal{H}_{SU(N)}$ ought to take the form

$$|\Psi^{(i)}\rangle = (-1)^i \mu_+^{a_1} \mu_+^{a_2} \cdots \mu_+^{a_i} \Phi_{a_1 a_2 \dots a_i}(\phi)|0\rangle, \tag{4.36}$$

where $1 \leq i \leq N(N-1)/2$.

One can proceed to compute $\mathcal{H}_G$ for any G in a similar manner. For brevity, we shall leave this to the interested reader.

4.2. The Nonperturbative Partition Function

We shall now furnish a general prescription that will allow us to compute, nonperturbatively, the partition function of our model on any three-manifold with target space X.

Suppose we have manifolds M_1 and M_2 whose boundaries are the same compact Riemann surface Σ but with opposite orientations, such that after gluing them along Σ, we get a new manifold M. Then, from the axioms of quantum field theory, the partition function on M with target space X would be given by

$$Z_X(M) = \langle M_2 | M_1 \rangle. \tag{4.37}$$

Here, $|M_1\rangle \in \mathcal{H}_1$ is a state due to the path integral over M_1 that is associated with Σ, and $|M_2\rangle \in \mathcal{H}_2$ is a state due to the path integral over M_2 that is also associated with Σ, where the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are canonically dual to each other.

We could also twist the boundary of M_1 by an element U of the mapping class group of Σ prior to gluing, whence the partition function on the resulting three-manifold M^U would be given by

$$Z_X(M^U) = \langle M_2|\widehat{U}|M_1\rangle, \tag{4.38}$$

where $\widehat{U}$ is an operator acting in $\mathcal{H}_1$ that represents U. In this manner, one can, with appropriate choices of Σ, M_1 and M_2, construct any three-manifold M^U, and upon determining how $\widehat{U}$ acts on $|M_1\rangle$ to produce another state in $\mathcal{H}_1$, the corresponding partition function on M^U can be determined via a tractable calculation on M. Therefore, let us determine the action of $\widehat{U}$ on $|M_1\rangle$.

For concreteness, let us consider $\Sigma = \mathbf{T}^2$ whence U is an element of $SL(2,\mathbb{Z})$, and M_1 is a solid torus. Then, we can conveniently choose on Σ, basic one-forms $\xi^{1,2}$ and basic one-cycles $C_{1,2}$, whereby

$$\int_{C_b} \xi^a = \delta_{ab}, \tag{4.39}$$

so that the matrix

$$U = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z}), \quad ps - qr = 1, \tag{4.40}$$

transforms the pair of cycles $(C_1, \ C_2)$ as

$$U : \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{4.41}$$

Now, from (4.17), we have

$$|M_1\rangle = |\psi_0\rangle \otimes \tilde{\Phi}_1|0\rangle, \tag{4.42}$$

where the subscripts '0' and '1' accompanying $|\psi\rangle$ and $\tilde{\Phi}$ are just convenient labels to associate them to $|M_1\rangle$.

Let us first determine how $\widehat{U}$ acts on $|\psi_0\rangle$. From the explanation of CS theory in [1], since $|\psi_0\rangle$ is associated with a path integral on M_1 with no operator insertions (see (4.18)), we can regard it as a vector v_0 in the Verlinde basis of $\mathcal{H}_{\mathrm{CS}}(A_0^\vartheta, \mathbf{T}^2)$, the space of integrable representations of the affine algebra associated with G at level k_{cs}, where the subscript '0' in v_0 means that it is associated with the trivial representation of G [1, section 4.3]. As such, according to *loc. cit.*, we have

$$\widehat{U}|\psi_0\rangle = K_0{}^j|\psi_j\rangle, \tag{4.43}$$

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where $|\psi_j\rangle$ corresponds to the vector v_j in $\mathcal{H}_{\mathrm{CS}}(A_0^\vartheta, \mathbf{T}^2)$ that is associated with the R_j representation of G; the R_j's are in one-to-one correspondence with the highest weights of G; K is the Verlinde matrix [14]; and the underlying wave function is

$$\psi_j(A_0^\vartheta) = \int_{A|_{\mathbf{T}^2}=A_0^\vartheta} DA \, W_j(C) \, e^{-S_{cs}}, \tag{4.44}$$

where

$$W_j(C) = \mathrm{Tr}_{R_j} P \exp\left(\oint_C A^a \mu_{+a}\right) \tag{4.45}$$

is the Q-closed (and therefore gauge-invariant) trace of the holonomy of the one-form $A^a \mu_{+a}$ along the longitudinal cycle C in M_1 taken in the representation R_j.[6]

Next, let us determine the action of $\widehat{U}$ on $\tilde{\Phi}_1$, where $\tilde{\Phi}_1$ takes the generic form in (4.27). As $\eta^{\bar{I}}$ is a (geometrically-trivial) scalar on Σ, we only need to consider the action on χ_α^I.

Recall that we can write

$$\chi_\mu^I = \chi_\alpha^I \omega_\mu^\alpha(A_0^\vartheta), \tag{4.46}$$

where the ω^α's are covariant harmonic one-forms on Σ. This is similar to the case in RW theory, except that here, the ω^α's also depend on a certain flat connection background A_0^ϑ. Since we are free to choose A_0^ϑ, let us choose a background whereby there are two covariant harmonic forms on Σ, i.e., there are two solutions to (4.3). Then, the fermionic zero modes can be expressed as

$$\chi_\beta^I = \int_{C_\beta'} \chi_\alpha^I \omega^\alpha, \quad \alpha, \beta = 1, 2, \tag{4.47}$$

since

$$\int_{C_\beta'} \omega^\alpha = \delta_\beta^\alpha, \tag{4.48}$$

where C_β' are a pair of covariant basic one-cycles in $\mathbf{T}^2$. Assuming that our background is also such that $C_{1,2}'$ is not deformed away from $C_{1,2}$, we also have

$$U : \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}. \tag{4.49}$$

---

[6]Note that the arguments in [1, section 4.3] involve the Wilson loop operator $\mathscr{W}_j(C) = \mathrm{Tr}_{R_j} P \exp\left(\oint_C A^a T_a\right)$ and not $W_j(C) = \mathrm{Tr}_{R_j} P \exp\left(\oint_C A^a \mu_{+a}\right)$, where the T_a's are generators of the Lie algebra $\mathfrak{g}$ of G. Nevertheless, recall that $d\mu_{+a} = -i_{V_a}(\Omega)$, where the vector fields V_a associated with the G-action on X are generators of $\mathfrak{g}$; in other words, like the T_a's, the μ_{+a}'s can be labeled by representations of G. Also, under a gauge transformation with parameter Λ, we have $\delta_\Lambda(\mu_{+a}) = -f_{ac}^d \Lambda_d \mu_+^c$ and $\delta_\Lambda T_a = f_{ac}^d \Lambda_d T^c$. Last but not least, we have the Poisson bracket relation (2.2). Altogether, this means that we can, for all our purposes, regard μ_{+a} as the matrix T_a whence we can also regard W_j as a Wilson loop operator $\mathscr{W}_j$.

Then, according to (4.48) and (4.47),

$$\widehat{U} : \begin{pmatrix} \chi_1^I \\ \chi_2^I \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \chi_1^I \\ \chi_2^I \end{pmatrix}, \tag{4.50}$$

where we shall regard χ_2^I to be the annihilation operator. Therefore,

$$\widehat{U} : \tilde{\Phi}_1(\phi, \eta, \chi_1^I) \rightarrow \tilde{\Phi}_1(\phi, \eta, p\chi_1^I + q\chi_2^I). \tag{4.51}$$

We would like to emphasize that the modular transformation will not modify the intrinsic definition of the zero modes χ, ϕ and η (which depends on the respective differential operators covariant with respect to A_0^ϑ).[7] Hence, the (generic) intrinsic definition (4.27) of $\tilde{\Phi}_1$ will not be modified either. Thus, there is no ambiguity in the map (4.51).

At any rate, the property that the vacuum $|0\rangle$ would be annihilated by χ_2^I must also hold after a transformation by $\widehat{U}$; in other words, if

$$\chi_2^I|0\rangle = 0, \tag{4.52}$$

then

$$\widehat{U} : |0\rangle \rightarrow |0\rangle', \tag{4.53}$$

where

$$(r\chi_1^I + s\chi_2^I)|0\rangle' = 0. \tag{4.54}$$

Since (4.4) means that we can represent χ_1^I by multiplication and χ_2^I by $\Omega^{IJ}\partial/\partial\chi_1^J$, we can also write (4.54) as

$$\left(s\Omega^{IJ}\frac{\partial}{\partial\chi_1^J} + r\chi_1^I \right)|0\rangle' = 0, \tag{4.55}$$

which implies that

$$|0\rangle' = \exp\left(-\frac{r}{s}\Omega_{IJ}\chi_1^I\chi_1^J \right)|0\rangle. \tag{4.56}$$

---

[7]This claim can be justified as follows. First, note that A_0^ϑ, being flat, is a covariant harmonic one-form on Σ, just like χ^I; hence, its components A_{01}^ϑ and A_{02}^ϑ will transform as in (4.50). Nevertheless, the covariant equation $\Omega_{IJ}\epsilon^{\tau\mu\nu}\partial_\mu\chi_\nu^J + \Omega_{IK}\epsilon^{\tau\mu\nu}\partial_J V_a^K A_{0\mu}^a\chi_\nu^J = 0$; $\mu,\nu = 1,2$, which defines χ, is a scalar equation on Σ (since the μ and ν indices are fully contracted) – it is thus insensitive to the transformation of Σ by U whence the definition of χ would be unmodified. As for ϕ and η, they are defined by the following one-form equations on Σ (since there is a free μ index): $\partial_\mu\phi + A_\mu^a V_a = 0$ and $g_{I\bar{J}}\partial^\mu\eta^{\bar{J}} + g_{I\bar{K}}\partial_{\bar{J}} V_a^{\bar{K}} A_0^{a\mu}\eta^{\bar{J}} = 0$. If we rewrite these equations as $D_\mu\phi = 0$ and $\mathcal{D}^\mu\eta = 0$, $\mu = 1,2$, then the action of U on Σ would map the first equation from $D_1\phi = 0 \rightarrow pD_1\phi + qD_2\phi = 0$ and $D_2\phi = 0 \rightarrow rD_1\phi + sD_2\phi = 0$. But $D_1\phi$ and $D_2\phi$ are independent quantities whence $pD_1\phi + qD_2\phi = 0$ and $rD_1\phi + sD_2\phi = 0$ imply that $D_\mu\phi = 0$, $\mu = 1,2$, which is the same as the original equation. The same argument applies for the second equation involving η. Hence, the definition of ϕ and η would also be unmodified.

In all, this means that $\widehat{U}$ acts on $|M_1\rangle$ as

$$\widehat{U} : |M_1\rangle \to |M_1\rangle', \tag{4.57}$$

where

$$|M_1\rangle' = K_0{}^j |\psi_j\rangle \otimes \tilde{\Phi}_1 \left(\phi, \eta, p\chi_1^I + q\Omega^{IJ} \frac{\partial}{\partial \chi_1^J} \right) \cdot \exp \left(-\frac{r}{s} \Omega_{IJ} \chi_1^I \chi_1^J \right) |0\rangle. \tag{4.58}$$

We are now ready to compute the partition function $Z_X(M^U)$. If

$$\langle M_2| = \langle 0| \tilde{\Phi}_2^\dagger (\phi, \eta, \chi_1^I) \otimes \langle \gamma|, \tag{4.59}$$

then $Z_X(M^U) = \langle M_2 | \widehat{U} | M_1 \rangle = \langle M_2 | M_1 \rangle'$ can also be expressed as

$$Z_X(M^U) = K_0{}^j \langle \gamma | \psi_j \rangle \cdot \langle 0 | \tilde{\Phi}_2^\dagger (\phi, \eta, \chi_1^I) \, \tilde{\Phi}_1 (\phi, \eta, p\chi_1^I + q\Omega^{IJ}\partial/\partial\chi_1^J) \exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle. \tag{4.60}$$

Notice that $\langle \gamma | \psi_j \rangle$ is just the CS path integral on M with an insertion of the Wilson loop operator $W_j(C)$ along the trivial knot C, i.e., it is the topologically-invariant expectation value $\langle W_j(C)\rangle_{\mathrm{CS}}$ of $W_j(C)$ in CS theory on M (see also footnote 6).

Notice also that $\langle 0 | \tilde{\Phi}_2^\dagger (\phi, \eta, \chi_1^I) \, \tilde{\Phi}_1 (\phi, \eta, p\chi_1^I + q\Omega^{IJ}\partial/\partial\chi_1^J) \exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle$ can be expressed as the scalar product $\langle \Phi^{(2)} | \Phi^{(1)} \rangle$ of the non-dynamically gauged RW theory on M with target X, where $|\Phi^{(2)}\rangle = \tilde{\Phi}_2(\phi, \eta, \chi_1^I)|0\rangle$, and $|\Phi^{(1)}\rangle = \tilde{\Phi}_1(\phi, \eta, p\chi_1^I + q\Omega^{IJ}\partial/\partial\chi_1^J) \cdot \exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle = \tilde{\Phi}_1'(\phi, \eta, \chi_1^I)|0\rangle$. Since $\tilde{\Phi}_1'(\phi, \eta, \chi_1^I)$ and $\tilde{\Phi}_2(\phi, \eta, \chi_1^I)$ correspond to classes in $H_G(\mathcal{M}^\vartheta)$, the scalar product $\langle \Phi^{(2)} | \Phi^{(1)} \rangle = \langle 0 | \tilde{\Phi}_2^\dagger \tilde{\Phi}_1'|0\rangle$ can be computed via G-equivariant Poincáre duality as an intersection number $(\tilde{\Phi}_2, \tilde{\Phi}_1')_{\mathcal{M}^\vartheta}$ of G-equivariant cycles in $\mathcal{M}^\vartheta$ that are dual to $\tilde{\Phi}_2$ and $\tilde{\Phi}_1'$, respectively.

Therefore, we can actually write

$$\boxed{Z_X(M^U) = K_0{}^j \langle W_j(C) \rangle_{\mathrm{CS}(M)} \cdot (\tilde{\Phi}_2, \tilde{\Phi}_1')_{\mathcal{M}^\vartheta(M,X)}} \tag{4.61}$$

As claimed, the partition function on M^U can be calculated in terms of well-defined quantities on M. In fact, it can be expressed as a product of a CS and an equivariant RW topological invariant of M!

5. New Knot Invariants From Supersymmetric Wilson Loops

As mentioned in subsection 4.2, a Q-invariant (and therefore gauge-invariant) Wilson loop operator along a knot $\mathcal{K} \subset M$ can be constructed as[8]

$$W_R(\mathcal{K}) = \mathrm{Tr}_R P \exp\left(\oint_{\mathcal{K}} A^a \mu_{+a}\right), \tag{5.1}$$

where M is an arbitrary Riemannian three-manifold, and R denotes the representation R of the Lie group G which acts on the hyperkähler target space X. The trace Tr is taken over G whose Lie algebra $\mathfrak{g}$ is generated by the μ_{+a}'s in the representation R (see footnote 6).

The Canonical Formalism

In the canonical formalism of section 4, where we restrict ourselves to the zero modes of the fields in the region $\Sigma \times I \subset M$ of interest, we have, in the absence of the Wilson loop operator $W_R(\mathcal{K})$, the "Gauss Law" constraint $\delta L/\delta A_\tau = 0$:

$$F^a_{\mu\nu} = 0, \tag{5.2}$$

where τ and $\{\mu, \nu\}$ are the coordinates on I and Σ, respectively.

If we include in our theory, multiple copies of the Wilson loop operator $W_{R_1}(\mathcal{K}_1) W_{R_2}(\mathcal{K}_2) \cdots$ in the representations $R_1, R_2, \ldots$ of G, the "Gauss Law" constraint becomes

$$F^a_{\mu\nu} = \epsilon_{\mu\nu} \sum_s \delta^2(x - P_s) \mu^a_{+(s)}. \tag{5.3}$$

Here, the P_s's are the points on Σ that the knots $\mathcal{K}_1, \mathcal{K}_2, \ldots$ intersect, and they are labeled by the representations R_s via the $\mu^a_{+(s)}$'s (see footnote 6).

The physical Hilbert space $\mathcal{H}_{\Sigma, P_s, R_s}$ of our theory can then be obtained by quantizing the underlying symplectic phase space $\mathcal{M}$ determined by the $A_\mu, \eta, \chi_\tau, \chi_\mu$ fields, their respective momentum conjugate A_ν, χ_τ, η, χ_ν (computed in (4.2)), the constraint (5.3), and the conditions $D\phi = 0$, $D\chi = 0$ and $D\eta = 0$. Specifically, according to the theory of geometric quantization [15], $\mathcal{H}_{\Sigma, P_s, R_s}$ would correspond to the space $H^0(\mathcal{L}, \mathcal{M})$ of holomorphic sections of a certain line bundle $\mathcal{L}$, where the curvature of $\mathcal{L}$ is given by $\sqrt{-1}$ times the symplectic two-form of $\mathcal{M}$.

The Path Integral Formalism and New Knot Invariants

Now that we have furnished, through the canonical formalism perspective, a formal description of the physical Hilbert space of the theory in the presence of multiple Wilson

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[8]Note that we have used the relation $\mu^a_+ \mu_{+a} = 0$ to construct the following expression.

loop operators $W_{R_j}(\mathcal{K}_j)$, let us compute explicitly the expectation value of such Wilson loop operators which will provide us with *new* knot invariants of three-manifolds.

To compute the expectation value $\langle W_R(\mathcal{K})\rangle$ via the path integral, we will need to replace $W_R(\mathcal{K})$ in (5.1) with its gauge-fixed version. To this end, recall that after gauge-fixing, Q would be replaced by $\widehat{Q} = Q + Q_{\mathrm{FP}}$. Of course, $W_R(\mathcal{K})$ in (5.1) is no longer invariant under the field transformations generated by $\widehat{Q}$. Nevertheless, a $\widehat{Q}$-invariant gauge-fixed replacement can be constructed as

$$\boxed{\tilde{W}_R(\mathcal{K}) = \mathrm{Tr}_R P \exp \oint_{\mathcal{K}} \mathbb{A} := \mathrm{Tr}_R P \exp \left(\oint_{\mathcal{K}} (A_a \mu_+^a + \chi^J \partial_J \mu_+^a c_a - \frac{1}{2} f_{abd} A^a c^b c^d) \right)} \quad (5.4)$$

One can show that

$$\delta_{\widehat{Q}} \oint \left(A_a \mu_+^a + \chi^J \partial_J \mu_+^a c_a - \frac{1}{2} f_{abd} A^a c^b c^d \right) = \oint d \left(c_a \mu_+^a - \frac{1}{6} f_{abd} c^a c^b c^d \right) = 0, \quad (5.5)$$

so

$$\delta_{\widehat{Q}} \tilde{W}_R(\mathcal{K}) = 0, \quad (5.6)$$

as claimed.

To compute perturbatively the following expectation value of multiple Wilson loops

$$\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle = \int D\phi DA D\eta D\chi Dc D\bar{c}\ e^{-S} \prod_j \tilde{W}_{R_j}(\mathcal{K}_j), \quad (5.7)$$

(where the Lagrange multiplier field B has already been integrated out to give the gauge-fixing condition $\partial^\mu A_\mu^a = 0$), we first expand each $\tilde{W}_j$ around the flat connection A_0 and the covariantly constant map ϕ_0 as

$$\tilde{W}_{R_j}(\mathcal{K}_j) = \mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_j} \mathbb{A} = \mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_i} \left\{ A_{0a} \mu_+^a(\phi_0) + \tilde{A}_a \mu_+^a(\phi_0) + \tilde{A}_a \partial_J \mu_+^a(\phi_0) \varphi^J + \cdots \right.$$
$$+ \chi^J \partial_J \mu_+^a(\phi_0) c_a + \chi^J \partial_K \partial_J \mu_+^a(\phi_0) \varphi^K c_a + \cdots$$
$$\left. - \frac{1}{2}(f_{abd} A_0^a c^b c^d + f_{abd} \tilde{A}^a c^b c^d) \right\}, \quad (5.8)$$

where $\tilde{A}$ and φ are fluctuations around A_0 and ϕ_0, and '$\cdots$' denotes all other expansion

terms around ϕ_0. Notice that the above path-ordered exponential can also be expressed as

$$\mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_j} \mathbb{A} = \mathrm{Tr}_{R_j} \left(\exp \int_{\mathcal{K}_j} A_0 \mu_+(\phi_0) \right)$$
$$\times [1$$
$$+ \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j} \left(\tilde{A}\mu_+(\phi_0) + \tilde{A}\partial_J\mu_+(\phi_0)\varphi^J + \chi^J\partial_J\mu_+(\phi_0)c - \frac{1}{2}(A_0cc + \tilde{A}cc) + \cdots \right)$$
$$+ \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j \times \mathcal{K}_j} \left(\tilde{A}\mu_+(\phi_0) + \tilde{A}\partial_J\mu_+(\phi_0)\varphi^J + \chi^J\partial_J\mu_+(\phi_0)c - \frac{1}{2}(A_0cc + \tilde{A}cc) + \cdots \right)^2$$
$$+ \cdots], \tag{5.9}$$

where we have and shall henceforth omit the Lie algebra index for notational simplicity. Note that because of (2.3), any term in the correlation function which contains $\mathrm{Tr}_{R_j}\left(\mu_+^2(\phi_0)\right)$ is automatically zero.[9]

After performing the expansion, we can evaluate the correlation function of $\prod_j \tilde{W}_{k_j}(\mathcal{K}_j)$ by the same method used to evaluate the partition function in section 3. Because of (5.9), we can, like in (3.57), express the correlation function as

$$\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle = \sum_{A_0^\vartheta} \left(e^{-\int_M k_{cs}L_{cs}(A_0^\vartheta)} \cdot Z_0(A_0^\vartheta) \cdot \prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)} \right) \mathfrak{W}(M, X, G; A_0^\vartheta; k_{cs}),$$
$$\tag{5.11}$$

where the first factor in parenthesis is manifestly topologically-invariant $- e^{-\int_M k_{cs}L_{cs}(A_0^\vartheta)}$ is the topological factor coming from the Chern-Simons part of the total Lagrangian evaluated at a flat connection A_0^ϑ, $Z_0(A_0^\vartheta)$ is the topological one-loop contribution given in (3.28), and $\prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)}$ is the product of noninteracting topological Wilson loops – and

$$\mathfrak{W}(M, X, G; A_0^\vartheta; k_{cs}) = \sum_\Gamma \mathfrak{W}_\Gamma(M, X, G; A_0^\vartheta; k_{cs}^m). \tag{5.12}$$

Here, $\sum_\Gamma$ is a sum over all possible Feynman diagrams with two or more loops that (i) have the right number of fermionic zero modes to absorb those that appear in the path integral measure, and (ii) are free of the coupling constant k. The label k_{cs}^m (where m may vanish) means that Γ carries with it a factor of k_{cs}^m.

Note that we have two types of Feynman diagrams here. The first type is where the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ do not contract with the vertices of the Lagrangian L; let us denote this

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[9]For example, according to (2.3), the correlation function

$$\langle \mathrm{Tr}_{R_j}(\tilde{A}\mu_+(\phi_0)\tilde{A}\mu_+(\phi_0)) \rangle \sim \langle \mathrm{Tr}_{R_j}(\mu_+^2(\phi_0)) \rangle = 0 \tag{5.10}$$

for the classical configuration ϕ_0.

type of diagrams as Γ^*. The second type is where the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ contract with the vertices of L; let us denote this type of diagrams as $\Gamma^\diamond$. In other words, we can write the total expectation value as

$$\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle = \langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^*} + \langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}. \tag{5.13}$$

Because the total expectation value $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle$ is topologically-invariant at the outset, we have

$$\frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle}{\delta h_{\mu\nu}} = \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}}{\delta h_{\mu\nu}} + \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}}{\delta h_{\mu\nu}} = 0, \tag{5.14}$$

where $h_{\mu\nu}$ is the metric of M.

Similar to CS and RW theory, because the propagators are not topologically-invariant, each diagram in $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle$ is not topologically-invariant by itself. However, the total expectation value is still topological because the variations (under a change in $h_{\mu\nu}$) of the diagrams cancel themselves out exactly.

In our case, notice that for the diagrams Γ^*, we only have the propagator factors

$$\int_{M \times M} d^3x_l d^3y_l \; \Delta(x_l, y_l) \tag{5.15}$$

and

$$\int_{\mathcal{K}_i \times \mathcal{K}_j} dx_l dy_l \; \Delta(x_l, y_l), \tag{5.16}$$

while for the diagrams $\Gamma^\diamond$, we *also* have the propagator factor

$$\int_M d^3x_l \int_{\mathcal{K}_j} dy_l \; \Delta(x_l, y_l) \tag{5.17}$$

coming from the contractions between the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ and that of the Lagrangian L. This means that the variations of the Γ^* diagrams cannot cancel out the variations of the $\Gamma^\diamond$ diagrams. In turn, this and (5.14) imply that

$$\frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}}{\delta h_{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}}{\delta h_{\mu\nu}} = 0 \tag{5.18}$$

simultaneously. In other words, both $\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^*}$ and $\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}$ are *independently* topologically-invariant.

For brevity, let us henceforth focus our discussion on $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}$. For the diagrams Γ^*, we can write

$$\mathfrak{W}_{\Gamma^*}(M, X, G; A_0^\vartheta; k_{cs}^m) = \int_{\mathcal{M}^\vartheta} \sqrt{g} \, d^{2n}\phi_0^I d^{2n}\phi_0^{\bar{I}} \, W_{\Gamma^*}(X, G; \phi_0, A_0^\vartheta) f_{\Gamma^*}(X, G; \phi_0, A_0^\vartheta) I'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m) \prod_j \Gamma^*_{\tilde{W}_j},$$

$$\tag{5.19}$$

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where the functions W_{Γ^*}, f_{Γ^*} and I'_{Γ^*} are similar to those in (3.70)–(3.71) as they result solely from contractions among the vertices coming from L, and $\Gamma^*_{\tilde{W}_j}$ denotes the contribution of $\tilde{W}_{R_j}(\mathcal{K}_j)$ to each $\mathfrak{W}_{\Gamma^*}$.

According to the discussion leading up to (3.69), we can also write[10]

$$\Gamma^*_{W_j} = W_{\Gamma^*, W_j}(X, G; \phi_0, A_0^\vartheta)\, f_{\Gamma^*, W_j}(X, G; \phi_0, A_0^\vartheta)\, I'_{\Gamma^*, W_j}(M; A_0^\vartheta). \tag{5.22}$$

Thus, we have

$$\mathfrak{W}_{\Gamma^*}(M, X, G; A_0^\vartheta; k_{cs}^m) = \mathscr{W}_{\Gamma^*}(X, G; A_0^\vartheta)\, \mathscr{I}'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m), \tag{5.23}$$

where

$$\boxed{\mathscr{W}_\Gamma(X, G; A_0^\vartheta) = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi_0^I\, d^{2n}\phi_0^{\bar{I}}\, \Big(W_{\Gamma^*} f_{\Gamma^*} \prod_j W_{\Gamma^*, W_j} f_{\Gamma^*, W_j}\Big)(X, G; \phi_0, A_0^\vartheta)} \tag{5.24}$$

can be regarded as a weight factor which combines the structural information of the hyperkähler manifold X and the Lie algebra $\mathfrak{g}$ of the gauge group G, and

$$\boxed{\mathscr{I}'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m) = \Big(I'_{\Gamma^*} \prod_j I'_{\Gamma^*, W_j}\Big)(M; A_0^\vartheta; k_{cs}^m)} \tag{5.25}$$

Therefore,

$$\Big\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j)\Big\rangle_{\Gamma^*} = \sum_{A_0^\vartheta} \sum_{\Gamma^*} \Big(e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)} \cdot Z_0 \cdot \prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)}\Big)\, \mathscr{W}_{\Gamma^*}(X, G; A_0^\vartheta)\, \mathscr{I}'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m),$$

$$\tag{5.26}$$

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[10]For example, consider the contribution $\Gamma^*_{\tilde{W}_j, A\varphi}$ from the term

$$\mathrm{Tr}_{R_j} \int_{\mathcal{K}_j \times \mathcal{K}_j} \tilde{A}\partial_I \mu_+ \varphi^I \tilde{A}\partial_J \mu_+ \varphi^J \tag{5.20}$$

in (5.9) (assuming that the fermionic zero modes in the measure have been absorbed exactly by the vertices from L which accompany this term). Performing the contraction, we get

$$\begin{aligned}
\Gamma^*_{\tilde{W}_j, A\varphi} &= \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j \times \mathcal{K}_j} dx^\mu dx^\nu\, \partial_I \mu_+ \partial_J \mu_+ \overbrace{\tilde{A}_\mu \tilde{A}_\nu} \varphi^I \varphi^J \\
&= \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j \times \mathcal{K}_j} dx^\mu dx^\nu\, \partial_I \mu_+ \partial_J \mu_+ \Delta_\mu^{(A\varphi)I} \Delta_\nu^{(A\varphi)J} \\
&= \mathrm{Tr}_{R_j} \Big(\partial_I \mu_+ \partial_J \mu_+\, f^I(X, G) f^J(X, G) \int_{\mathcal{K}_j \times \mathcal{K}_j} dx^\mu dx^\nu \Delta_\mu^{'(A\varphi)} \Delta_\nu^{'(A\varphi)}\Big) \\
&= W_{\Gamma^*_{W_j}, A\varphi}(X, G; \phi_0, A_0^\vartheta)\, f_{\Gamma^*_{W_j}, A\varphi}(X, G; \phi_0, A_0^\vartheta)\, I'_{\Gamma^*_{W_j}, A\varphi}(M; A_0^\vartheta),
\end{aligned} \tag{5.21}$$

where μ, ν run over all directions in M, and the second-last equality is due to (3.46).

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where

$$\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^*, A_0^\vartheta} = \sum_{\Gamma^*} \mathscr{W}_{\Gamma^*}(X, G; A_0^\vartheta) \, \mathscr{I}'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m) \qquad (5.27)$$

is a *new* knot invariant of three-manifolds which depends on both G and X, that also defines a *new* knot weight system whose weights $\mathscr{W}_{\Gamma^*}(X, G; A_0^\vartheta)$ are characterized by both Lie algebra structure *and* hyperkähler geometry.

References

[1] E. Witten, "Quantum Field Theory And The Jones Polynomial", Commun. Math. Phys. **121** (1989) 351.

[2] V.F.R. Jones, "A polynomial invariant for knots via von Neumann algebra". Bull. Amer. Math. Soc. (N.S.) **12**: 103–111, 1985.

[3] D. Bar-Natan, "Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory", Ph.D. thesis, 109 pp, Princeton Univeristy June 1991.

[4] S. Axelrod and I. M. Singer, "Chern-Simons Perturbation Theory", Proc. XXth DGM Conference (New York, 1991) (S. Catto and A. Rocha, eds) World Scientific, 1992, 345, [arXiv:hep-th/9110056]; "Chern–Simons Perturbation Theory II", J. Diff. Geom. **39** (1994), 173-213, [arXiv:hep-th/9304087].

[5] E. Guadagnini, M. Martellini and M. Mintchev, "Perturbative Aspects of the Chern-Simons Field Theory", Phys. Lett. **B228** (1989) 489 [30].

[6] M. Kontsevich, "Feynman diagrams and low-dimensional topology", in Proceedings of the first European Congress of Mathematics, vol. 2, Progress in Math. **120**, Birkhäuser, Boston, 1994, 97-121.

[7] L. Rozansky and E. Witten, "Hyper-Kahler Geometry and Invariants of Three-Manifold", Selecta Mathematica, New Series. Vol **3**, Number 3, 401-458, [arXiv:hep-th/9612216].

[8] A. Kapustin and N. Saulina, "Chern-Simons-Rozansky-Witten topological field theory", Nucl. Phys. **B823**, Issue 3 (2009), 403-427, [arXiv:0904.1447].

[9] E. Koh, S. Lee, S. Lee, "Topological Chern-Simons sigma model", JHEP **0909** (2009) 122, [arXiv:0907.1641].

[10] J. Kallen, J. Qiu, M. Zabzine, "Equivariant Rozansky-Witten classes and TFTs", [arXiv:1011.2101].

[11] M. Mariño, *Chern-Simons theory, matrix models, and topological strings*, Oxford University Press, 2005; S. Elitzur, G. Moore, A. Schwimmer, N. Seiberg, "Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory", Nucl. Phys. **B326** (1989) 108.

[12] V. Guillemin and S. Sternberg, *Supersymmetry and Equivariant de Rham Cohomology*, Springer, Berlin (1999).

[13] M. Brion, private communication.

[14] E. Verlinde, "Fusion rules and modular transformations in 2d conformal field theory", Nucl. Phys. **B300**, 360 (1988).

[15] N. M. J. Woodhouse. "Geometric Quantization", (1991), Clarendon Press.